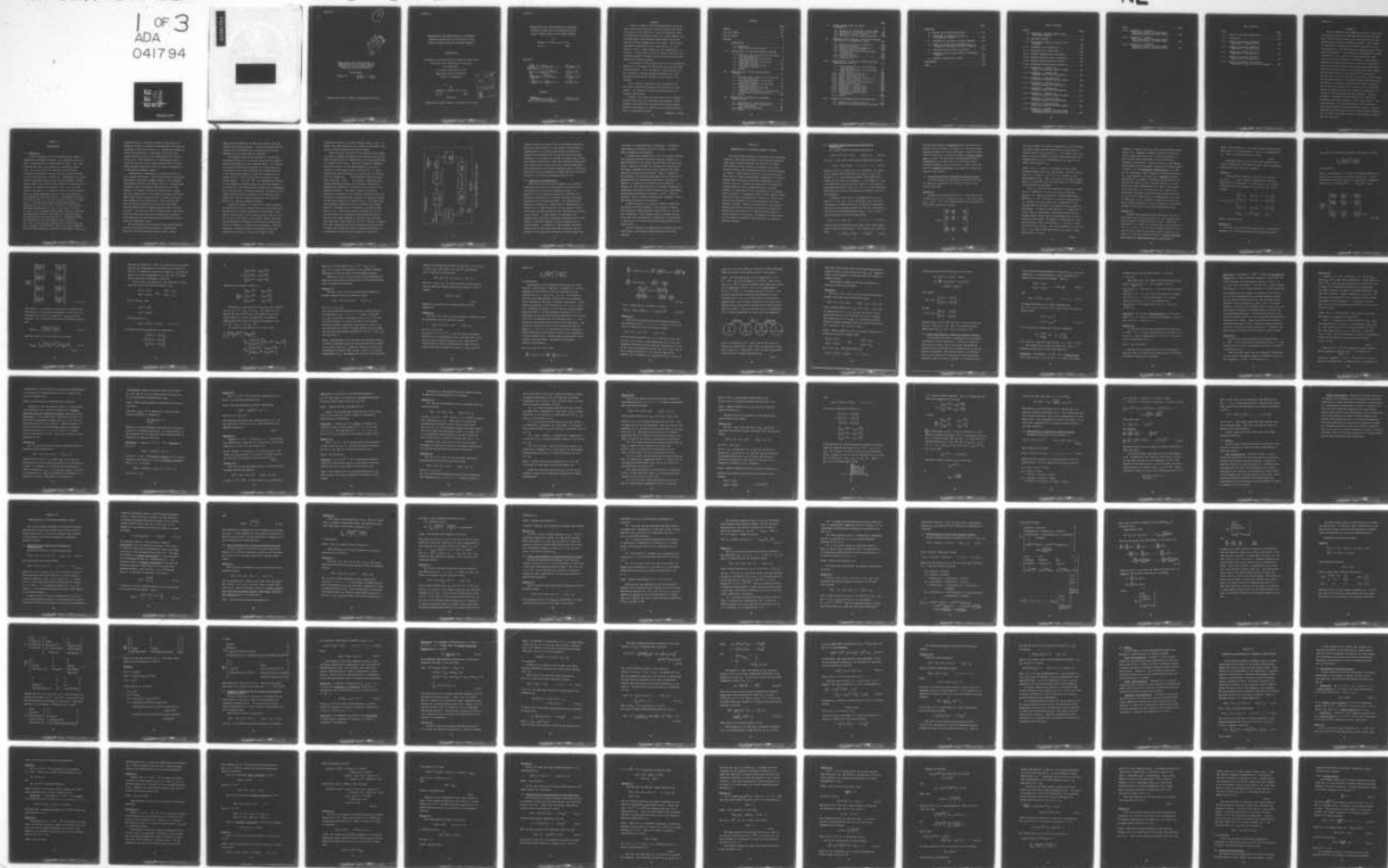AD-A041 794   AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OHIO SCH--ETC F/G 12/1
OBSERVABILITY AND IDENTIFIABILITY OF NONLINEAR DYNAMICAL SYSTEM--ETC(U)
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OBSERVABILITY AND IDENTIFIABILITY OF
NONLINEAR DYNAMICAL SYSTEMS WITH AN
APPLICATION TO THE OPTIMAL CONTROL
MODEL FOR THE HUMAN OPERATOR

DISSERTATION

DS/EE/77-1   Raymond E. Siferd
                    Lt Col   USAF

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(See form 1473)

OBSERVABILITY AND IDENTIFIABILITY OF NONLINEAR

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OPTIMAL CONTROL MODEL FOR THE HUMAN OPERATOR

DISSERTATION

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

by

Raymond E. Siferd, B.E.E., M.S.

Lt Col   USAF

June 1977

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PREFACE

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A vital part of this research involved the mathematical modeling of a simulator and the use of data from the simulator operation. I want to thank Dr. Grant McMillan and Mr. Tom Grace of Systems Research Laboratory for their outstanding cooperation and help on this aspect of the problem.

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Finally, my wife Sue and sons Mike and Jeff deserve a special thanks for their continued encouragement, understanding, and sacrifices throughout this entire program. Without their support, I could not have succeeded and for this I am extremely grateful.

Raymond E. Siferd

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ABSTRACT

Two very important and fundamental topics in the theory of dynamical systems are observability and identifiability. There has been limited work on the observability and identifiability of nonlinear dynamical systems. This problem is addressed by viewing the observation process as a nonlinear mapping of the initial states (and parameters) to the output measurements. The problem of observability and identifiability is then reduced to finding conditions under which the nonlinear mapping is one-to-one. Using such an approach, the theory of nonlinear functions is extended to find new sufficient conditions for local and global observability and identifiability for nonlinear dynamical systems. The nonlinear theory is then applied to the problem of determining the identifiability of the optimal control model for the human operator. The model assumes that the operator performance is dictated by the desire to behave optimally with respect to a chosen cost functional under constraints. Within the model structure are a number of parameters, the values of which determine the model response. Previous attempts to establish the identifiability of the model parameters have been limited to linear system theory. The identifiability of the model parameters is established using the previously developed nonlinear theory, a gradient computational technique is used to estimate model parameters, and the results of model response to experimental simulator data are presented.

Chapter I

INTRODUCTION

1.1 Background

In this research, we will be concerned with certain properties of dynamical systems. By a dynamical system, we mean a structure which at each moment of time $t \in [t_o, t_f]$ receives some input $u(t)$ and emits an observable output $y(t)$. It is noted that the important class of models referred to as dynamical systems are more formally defined in the literature (e.g., see Ref 17). Knowledge of the structure of a dynamical system allows one to formulate a mathematical model of the system which includes a state-transition function (or solution or trajectory) $g(t, x_o, u, \phi)$ whose value is the state $x(t)$ resulting at time t from the initial state x_o, system parameter vector ϕ, and input function u. It is noted that the state at time t is dependent not only on the current value of u, but also the past history of u from t_o to t. The observed output or observation process is described by $y(t) = h(t, x(t), \phi)$ for all $t \in [t_o, t_f]$ for continuous observations or $y(t_i) = h(t_i, x(t_i), \phi)$ with $t_i \in [t_o, t_f]$ for discrete observations. Two very important and fundamental topics in the theory of dynamical systems are observability and identifiability. A dynamical system is observable if, with knowledge of $y(t)$ and $u(t)$ on $[t_o, t_f]$, one can uniquely determine the initial state, x_o. If x_o is found, then the state vector $x(t)$ can be computed using the state-transition function for any time $t \in [t_o, t_f]$.

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Observability is an important property since control of dynamical systems is ordinarily implemented on the basis of knowledge of the system state $x(t)$. The related problem of system identifiability is concerned with determining a unique parameter vector ϕ with knowledge of $y(t)$ and $u(t)$ on $[t_o, t_f]$. Identifiability is a fundamental property since one wants to select a model structure and an input/output relation which will allow the unique determination of the unknown parameter values.

Conditions for observability of linear systems have been studied extensively. Some original efforts concerning observability were due to Kalman (Refs 18, 19), Kreindler and Sarachik (Ref 30), and Gilbert (Ref 10). Likewise many researchers have investigated identification of linear systems. For example, Bellman and Åstrom (Ref 6), Mehra (Ref 36), Denham (Ref 8), and others investigate conditions under which certain canonical forms of linear time invariant systems will be identifiable. Glover and Williams (Ref 11) develop conditions for both local and global identifiability from the system frequency domain transfer function. Tse and Anton (Ref 50), among others, have formulated necessary and sufficient conditions for assessing identifiability for linear systems with stochastic disturbances and observation noise.

The investigation of observability and identifiability of nonlinear systems has been limited. Kostyukouskii (Refs 28, 29) and Griffith and Kumar (Ref 17) report

observability conditions for nonlinear systems under continuous observation processes. Grewall and Glover (Ref 12) address local identifiability by linearizing the state equations in the neighborhood of interest.

The approach taken in this research is to view the observation process as a nonlinear mapping of the initial states and parameters to the observed output. For example, for a discrete observation process the initial states x_o are mapped via the state transition and output functions to the observation sequence, which may be denoted by an output vector Y. That is $Y(x_o)$ may be viewed as a nonlinear mapping, $Y : D \subset R^n \rightarrow R^m$. The observability problem is then reduced to finding conditions where the mapping Y is one-to-one. The identification problem can be approached in a similar manner. With this approach, the theory of nonlinear functions can be extended to find sufficient conditions for local and global observability and identifiability for nonlinear dynamical systems.

A practical example of a case where a model structure for dynamical systems is defined, but where the question of identifiability of model parameters is unresolved, is the optimal control model for the human operator (Refs 20, 21, 22, 23, 25). The optimal control model assumes that operator performance is dictated by the desire to behave optimally with respect to a chosen cost functional. Using the techniques of modern control and estimation theory and incorporating factors to account for inherent human

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limitations results in a model structure shown in Fig. 1.1. Within this model structure are a number of parameters, the values of which determine the model response.

Given the basic structure of the optimal control model, one needs to address the question of identifiability of the model parameters and, if identifiable, use experimental data to estimate their values. Previous attempts to establish the identifiability of the model parameters have been limited to linear system theory (Refs 40, 41). In this research the identifiability of the model will be analyzed using the theory for nonlinear dynamical systems as previously discussed. The nonlinear theory is required since, without simplifying assumptions, the model input/output mathematical description is nonlinear with respect to the unknown parameters. The model is used in a tracking function to correspond to an available simulator system and the output (or measured data) consists of ensemble averages of many time histories of tracking errors. The model mean and covariance predictions will be developed in a manner similar to Kleinman's work on modeling a tracking problem (Refs 20, 23). The use of ensemble averaged data as the measured output for establishing identifiability differs from the more usual approach of using the time histories from individual trials (Refs 44, 48, 49). However, assuming this type of experimental data is available and the system is identifiable under this type of observation process, it is appealing to adjust system parameters to match the

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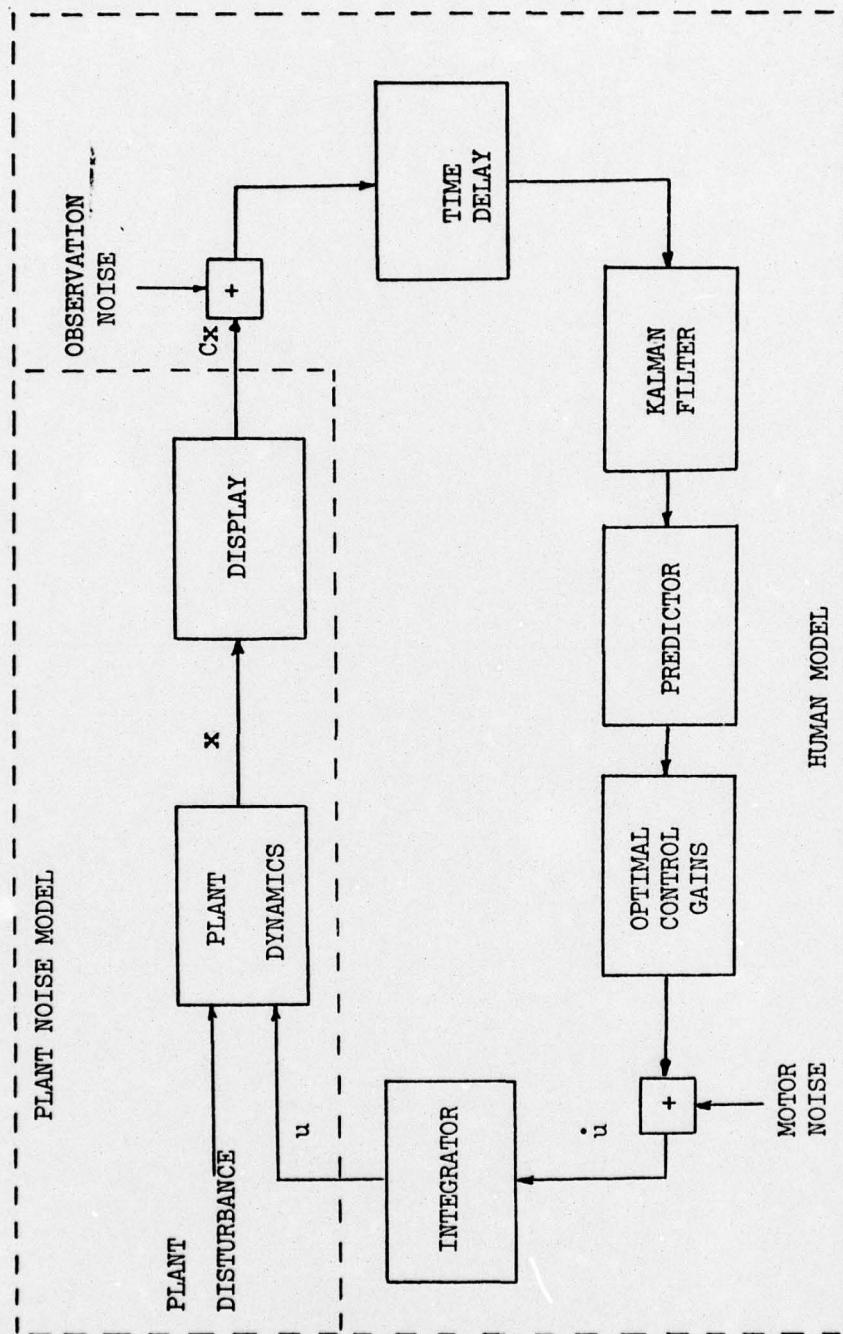
FIG. 1.1

STRUCTURE OF OPTIMAL CONTROL MODEL FOR HUMAN OPERATOR

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average response over many trials and different operators. After the identifiability of the optimal control model is established for a specified input/output arrangement, a computational procedure will be used to adjust the model parameters in an attempt to minimize the mean square difference between experimental output data and model output data. This differs from previous work where nominal values have been established for the model parameters using a subjective criteria to match experimental to model data and a heuristic procedure for adjusting parameters (Refs 20, 23).

1.2 Objectives and Organization

The first objective of this research is to develop sufficient conditions for establishing observability and identifiability of nonlinear dynamical systems. In Chapter II the observability question is addressed and sufficient conditions are developed for local and global observability of nonlinear systems. The identification problem is analyzed as a special case of observability in Chapter III, so that the results of this chapter include both local and global conditions for identifiability of non-linear systems. In both Chapter II and Chapter III, examples are presented to illustrate the application of the theory to nonlinear systems, as well as application to linear systems. In Chapter IV, some theorems concerning minimization of cost functionals are developed, and the relation of the problem of minimizing a least squares cost

functional to identifiability is presented. In addition,
computational procedures which can be used to estimate
model parameters are discussed.

A second major objective is to use the theory developed
for assessing nonlinear dynamical systems to address the
identifiability of the optimal control model for human oper-
ators. Chapter V develops the mathematical description of
the model, including the state space equations for propa-
gating mean states and covariances. Then in Chapter VI,
the simulator, from which experimental data is obtained, is
described and the mathematical description developed in
Chapter V is placed in the context of the simulator. In
Chapter VII, the identifiability of the model parameters is
established using the nonlinear theory and the input/output
relationships developed in Chapters V and VI. Also the
results of using a gradient computational technique to esti-
mate model parameters are presented.

A third objective is to provide further insight into
the modeling of human response. The research is unique
with respect to system dynamics, sensor displays, and con-
trol mechanisms. Other aspects, such as operator thresh-
olds, and other possible model refinements, are discussed
in Chapters V, VI, and VII as the particular aspect of the
model arises.

Finally, Chapter VIII summarizes the results and their
significance, and presents recommendations for further
research.

Chapter II

OBSERVABILITY OF NONLINEAR DYNAMICAL SYSTEMS

The observability problem is concerned with determining conditions under which knowledge of the input and observed output data uniquely determine the state of the system. This problem has been discussed extensively for linear systems; however, the investigation of the nonlinear problem is limited. Kostyukovskii (Refs 28, 29) and Griffith and Kumar (Ref 13) report observability conditions for nonlinear systems under continuous observation processes on a time interval $[t_o, t_f]$. Grewal and Glover (Ref 12) address the related problem of local identifiability of nonlinear systems by linearizing the state equations in the neighborhood of interest. The approach taken in this chapter differs from any of these investigations. Totally new results are obtained for determining the observability of nonlinear dynamical systems. Results are derived first for discrete observations and then extended to continuous observations. Examples are presented and the relation of the theory to linear systems is illustrated. In Chapter III, this theory will be applied to parameter identification in nonlinear dynamical systems.

2.1 Nonlinear State Equations and Definition of Observability

We consider nonlinear systems described by

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \qquad (2.1.1)$$

for all $t \in [t_o, t_f]$ and the discrete observation process

$$y(t_i) = h(t_i, x(t_i)) \qquad i = 1, 2, \cdots k \qquad (2.1.2)$$

where $x(t)$ is an n vector, $y(t)$ is an m vector, the input u is an r vector valued function of t, and $t_i \in [t_o, t_f]$. Assume $f(\cdot, \cdot, \cdot)$ has continuous partial derivatives with respect to its first two arguments. u is a member of a set U of admissible input functions, where $U \subset C$ the space of continuous functions on $[t_o, t_f]$. $h(\cdot, \cdot)$ is continuous and has continuous partial derivatives with respect to both of its arguments.

A solution to Eq (2.1.1) is denoted by $g(t, x_o, u)$ for all $t \in [t_o, t_f]$. For an admissible input u, we are interested in finding sufficient conditions to guarantee a one-to-one correspondence between the initial conditions x_o and the observed output sequence $\{y(t_i)\}$. $y(t_i)$ is an m vector denoting the discrete observations

$$y(t_i) = h(t_i, g(t_i, x_o, u)) \qquad i = 1, 2, \cdots k \qquad (2.1.3)$$

This observation process can be viewed as one large observation vector of dimension mk. Let Y denote this vector so that

$$Y = \left[y^T(t_1) \ y^T(t_2) \cdots y^T(t_k) \right]^T \qquad (2.1.4)$$

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The nonlinear system is <u>observable</u> under the observation process Y with the input u if there is a one-to-one correspondence between the initial conditions x_O and the observation vector $Y(x_O)$. We say the system is <u>locally</u> <u>observable</u> at x_O when the one-to-one correspondence holds in a neighborhood of x_O. When there is a one-to-one correspondence between x_O and Y for all $x_O \in X_O CR^n$, we say the system is <u>globally</u> <u>observable</u> on X_O under the observation process Y with input u.

2.2 <u>Local Observability of Nonlinear Dynamical Systems</u>

In this section, we will develop theorems which lead to sufficient conditions for a nonlinear system to be locally observable.

<u>Theorem 2.1</u>

Consider the nonlinear function $y = g(x)$ where y is a k vector and x is an n vector with $k \geq n$. This is denoted by $g : X CR^n \rightarrow R^k$. The Jacobian matrix will be denoted by $g'(x)$ and is defined as follows:

$$
g'(x) = \begin{bmatrix}
\dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\[2ex]
\dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial g_2}{\partial x_n} \\[1ex]
\vdots & & & \vdots \\[1ex]
\dfrac{\partial g_k}{\partial x_1} & \dfrac{\partial g_k}{\partial x_2} & \cdots & \dfrac{\partial g_k}{\partial x_n}
\end{bmatrix}
$$

Let x_0 be a point in X which is mapped by g to R^k and where g(x) has a strong Frechet derivative at x_0 and the k x n Jacobian matrix $g'(x_0)$ has rank n. If we denote $g(x_0)$ by y, then an inverse function g^{-1} exists which maps points in a neighborhood of y_0 to a neighborhood of x_0; that is g is locally one-to-one at x_0.

Proof: For k = n, this is a statement of the inverse function theorem (see, e.g., Ref 39:125). For this special case of k = n with $g : X \subset R^n \to R^n$, g is termed a local homeomorphism at x_0. Thus, at x_0 there is an open neighborhood $\delta \in X$ such that when g is restricted to δ, the mapping is one-to-one and continuous.

For k > n, if g'(x) has a rank of n, there is at least one n x n submatrix of $g'(x_0)$ (say $g_1'(x_0)$) such that $|g_1'(x_0)| \neq 0$. Thus we can define a locally homeomorphic mapping $g_1 : \delta \subset R^n \to R^n$ where δ is an open neighborhood of x_0. Further let $g_1(x_0)$ equal the n components of $g(x_0)$ corresponding to the rows of the submatrix $g_1'(x_0)$ which is nonsingular. If there is more than one n x n submatrix of $g'(x_0)$ which is nonsingular, arbitrarily choose the non-singular matrix incorporating the uppermost rows so that a unique g_1 is obtained. Now suppose g restricted to δ were not one-to-one. Then there exists $x_1 \in \delta$ and $x_2 \in \delta$ with $x_1 \neq x_2$ and $g(x_1) = g(x_2)$. But this contradicts g_1 being one-to-one at x_0 since by the definition of g_1, $g(x_1) = g(x_2)$ implies $g_1(x_1) = g_1(x_2)$.

Q.E.D.

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Theorem 2.1 requires that $g(x)$ have a strong Frechet derivative at x_o. Appendix A defines a Frechet derivative, a strong Frechet derivative, and a Gateaux-derivative, as well as some useful properties of these derivatives. Throughout this report, a derivative means in the Frechet sense except where the phrase "continuously differentiable" is used. If g is <u>continuously</u> <u>differentiable</u> on an open set D_o, this means g has a continuous Gateaux-derivative on D_o. From Appendix A, we note that a <u>continuous</u> Gateaux-derivative at x_o implies a continuous Frechet derivative. We also note that if g has a Frechet derivative at each point of an open neighborhood of x_o, then g' is strong at x_o if and only if g' is continuous at x_o. Thus in all sufficiency theorems, we could replace "strong derivative at x_o" by "has a derivative in a neighborhood of x_o which is continuous at x_o." However, we should note that g can have a strong derivative at x_o even though g is not differentiable at all points of an open neighborhood of x_o.

Theorem 2.2

Consider the nonlinear system described by Eq (2.1.1) with input u and initial condition x_o. The discrete observation process Y is defined by Eq (2.1.4). Assume the dimension of Y is equal to or greater than the dimension of $x(t)$; i.e., $mk \geq n$. If $Y(x)$ has a strong derivative at x_o and the rank of the $mk \times n$ Jacobian matrix $\partial Y(x_o)/\partial x_o$ is equal to n, then the nonlinear system is <u>locally</u> <u>observable</u> <u>at</u> <u>x_o</u> <u>under</u> <u>the</u> <u>observation</u> <u>process</u> <u>Y</u> <u>with</u> <u>input</u> <u>u</u>.

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Proof: From Theorem 2.1, an inverse function exists which
maps points in a neighborhood of $Y(x_o)$ to a neighborhood of
x_o. Thus Y is locally one-to-one at x_o.

Q.E.D.

Although Theorem 2.2 gives sufficient conditions for
local observability, the following lemma leads to conditions
which may be more practical to apply.

Lemma 2.1

A necessary and sufficient condition that vectors
x_1, x_2, $\cdots$, x_n be linearly independent is that their
Gram matrix be nonsingular. The Gram matrix G is defined
in terms of the inner products of x_i and x_j, (x_i, x_j) as
follows:

$$G = \left[(x_i, x_j) \right] = \begin{bmatrix} (x_1, x_1) & (x_1, x_2) \cdots (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) \cdots (x_2, x_n) \\ \vdots & \vdots \\ (x_n, x_1) & (x_n, x_2) \cdots (x_n, x_n) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_2 \cdots x_n \end{bmatrix}^T \begin{bmatrix} x_1 x_2 \cdots x_n \end{bmatrix}$$

Proof: See Ref 38:378.

Theorem 2.3

Let $Y(x_o)$ have a strong derivative at x_o. A sufficient
condition for a nonlinear system to be locally observable

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at x_o under the observation process Y with input u is that

$$\sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial x_o} \right]^T \left[\frac{\partial y(t_i)}{\partial x_o} \right]$$

be nonsingular.

Proof: From Theorem 2.2, we know a sufficient condition is that the rank of the mk x n matrix be equal to n. Thus it is sufficient that the columns of $\partial Y(x_o)/\partial x_o$ be independent. Recall that $Y = \left[y^T(t_1) y^T(t_2) \cdot \cdot \cdot y^T(t_k) \right]^T$, so that

$$\frac{\partial Y(x_o)}{\partial x_o} = \begin{bmatrix} \dfrac{\partial y(t_1)}{\partial x_{o1}} & \dfrac{\partial y(t_1)}{\partial x_{o2}} & \cdot \cdot \cdot & \dfrac{\partial y(t_1)}{\partial x_{on}} \\ \dfrac{\partial y(t_2)}{\partial x_{o1}} & \dfrac{\partial y(t_2)}{\partial x_{o2}} & \cdot \cdot \cdot & \dfrac{\partial y(t_2)}{\partial x_{on}} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \dfrac{\partial y(t_k)}{\partial x_{o1}} & \dfrac{\partial y(t_k)}{\partial x_{o2}} & \cdot \cdot \cdot & \dfrac{\partial y(t_k)}{\partial x_{on}} \end{bmatrix} \text{ mk x n}$$

$$(2.1.5a)$$

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$$\frac{\partial Y(x_o)}{\partial x_o} = \begin{vmatrix} \begin{bmatrix} \dfrac{\partial y_1(t_1)}{\partial x_{o1}} \\ \vdots \\ \dfrac{\partial y_m(t_1)}{\partial x_{o1}} \end{bmatrix} & \cdots & \begin{bmatrix} \dfrac{\partial y_1(t_1)}{\partial x_{on}} \\ \vdots \\ \dfrac{\partial y_m(t_1)}{\partial x_{on}} \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} \dfrac{\partial y_1(t_k)}{\partial x_{o1}} \\ \vdots \\ \dfrac{\partial y_m(t_k)}{\partial x_{o1}} \end{bmatrix} & \cdots & \begin{bmatrix} \dfrac{\partial y_1(t_k)}{\partial x_{on}} \\ \vdots \\ \dfrac{\partial y_m(t_k)}{\partial x_{on}} \end{bmatrix} \end{vmatrix}$$

(2.1.5b)

From Lemma 2.1, a necessary and sufficient condition that the columns of $\partial Y(x_o)/\partial x_o$ be independent is that the n x n Gram matrix of $\partial Y(x_o)/\partial x_o$ (denoted by $G_{\partial Y/\partial x_o}$) be non-singular. But recall from the definition of the Gram matrix that

$$G_{\partial Y/\partial x_o} = \left[\frac{\partial Y(x_o)}{\partial x_o} \right]^T \left[\frac{\partial Y(x_o)}{\partial x_o} \right] \tag{2.1.6}$$

Substituting Eq (2.1.5) into Eq (2.1.6) yields

$$G_{\partial Y/\partial x_o} = \sum_{i=1}^{k} \left[\frac{\partial y(t_i, x_o)}{\partial x_o} \right]^T \left[\frac{\partial y(t_i, x_o)}{\partial x_o} \right] \tag{2.1.7}$$

Q.E.D.

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Note that in Theorems 2.2 and 2.3 we have placed no require-
ments on the uniqueness of the solution of the system Eq
(2.1.1). Of course the solution as mapped by $Y(x_o)$ must be
unique in a local neighborhood of x_o, but this is assured
by the conditions of Theorems 2.2 and 2.3.

At this point, an example will be presented to illus-
trate the use of the previous theorems. Let

$$\dot{x}_1(t) = x_1(t) \qquad x_1(0) = x_{o1}$$
$$\dot{x}_2(t) = 2x_2(t) \quad \text{with} \quad x_2(0) = x_{o2}$$

for $t \in [0, t_f]$. Thus

$$x_1(t) = x_{o1}e^t$$

$$x_2(t) = x_{o2}e^{2t}$$

The observations are

$$y(t_i) = x_1^2(t_i) + x_2^2(t_i) \qquad i = 1, 2, 3$$

so that the observation process is described by

$$Y = \begin{bmatrix} x_1^2(t_1) + x_2^2(t_1) \\ x_1^2(t_2) + x_2^2(t_2) \\ x_1^2(t_3) + x_2^2(t_3) \end{bmatrix}$$

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or

$$
Y = \begin{bmatrix} x_{o1}{}^2 \, e^{2t_1} + x_{o2}{}^2 \, e^{4t_1} \\ x_{o1}{}^2 \, e^{2t_2} + x_{o2}{}^2 \, e^{4t_2} \\ x_{o1}{}^2 \, e^{3t_3} + x_{o2}{}^2 \, e^{4t_3} \end{bmatrix}
$$

Evaluating the Jacobian results in

$$
\frac{\partial Y(x_o)}{\partial x_o} = \begin{bmatrix} 2x_{o1} \, e^{2t_1} & 2x_{o2} \, e^{4t_1} \\ 2x_{o1} \, e^{2t_2} & 2x_{o2} \, e^{4t_2} \\ 2x_{o1} \, e^{2t_3} & 2x_{o2} \, e^{4t_3} \end{bmatrix}
$$

Therefore, we can state that $\partial Y(x_o)/\partial x_o$ has a rank of two for $x_o \, \epsilon \, \{R^2 \sim \{x_{o1} = 0 \text{ or } x_{o2} = 0\}\}$ where $\{A \sim B\}$ denotes the set of all points in A which are not in B. From Theorem 2.2, we can conclude that the system is locally observable under the observation process Y for $x_o \, \epsilon \, \{R^2 \sim \{x_{o1} = 0 \text{ or } x_{o2} = 0\}\}$.

If we were to use Theorem 2.3, we first form

$$
\sum_{i=1}^{k} \begin{bmatrix} \dfrac{\partial y(t_i, \, x_o)}{\partial x_o} \end{bmatrix}^T \begin{bmatrix} \dfrac{\partial y(t_i, \, x_1)}{\partial x_o} \end{bmatrix}
$$

$$
= \sum_{i=1}^{3} \begin{bmatrix} 2x_{o1} \, e^{2t_i} \\ 2x_{o2} \, e^{4t_i} \end{bmatrix} \begin{bmatrix} 2x_{o1} \, e^{2t_i} & 2x_{o2} \, e^{4t_i} \end{bmatrix}
$$

$$
= \sum_{i=1}^{3} \begin{bmatrix} 4x_{o1}{}^2 \, e^{4t_i} & 4x_{o1}x_{o2} \, e^{6t_i} \\ 4x_{o2}x_{o1} \, e^{6t_i} & 4x_{o2}{}^2 \, e^{8t_i} \end{bmatrix}
$$

Again this is nonsingular for $x_o \in \{R^2 \sim \{x_{o1} = 0 \text{ or } x_{o2} = 0\}\}$, so that from Theorem 2.3, the system is locally observable on this set under the observation process Y.

Theorems 2.2 and 2.3 can be used to make statements of sufficient conditions for observability of nonlinear systems under continuous observations.

Theorem 2.4

Let $y(t) = h(t, x(t))$ be the continuous observation process associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o$$

for all $t \in [t_o, t_f]$.

Let Y be the discrete observation process vector formed by the components $y(t_i)$, $i = 1, 2 \cdots k$ with $k \geq \frac{n}{m}$ where $t_i \in [t_o, t_f]$ are any k time points on $[t_o, t_f]$. If Y has a strong derivative at x_o and the rank of the Jacobian $\partial Y(x_o)/\partial x_o$ formed by any such k time points is equal to n, then the nonlinear system is locally observable at x_o under the continuous observation process $y(t)$ for all $t \in [t_o, t_f]$ with the input u.

Proof: From Theorem 2.2 we know that the nonlinear system is locally observable under the discrete observation process Y at x_o with input $u(t)$. Therefore, there is a one-to-one correspondence between $Y(x_o^\delta)$ and $x_o^\delta \in \delta$ where δ is an open neighborhood of x_o. Now suppose the system is not observable

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under the continuous observation process $y(t) = h(t, x(t))$;
$t \in [t_o, t_f]$. Then there is an x_o^1 and x_o^2, with x_o^1,
$x_o^2 \in \delta$ and $x_o^1 \neq x_o^2$, such that

$$h(t, g(t, x_o^1, u)) = h(t, g(t, x_o^2, u))$$

for all $t \in [t_o, t_f]$. But this contradicts local observ-
ability under the discrete observation process Y since it
implies

$$Y(x_o^1) = Y(x_o^2)$$

Q.E.D.

Similarly we can make the following statement from
Theorem 2.3.

Theorem 2.5

Let $y(t) = h(t, x(t))$ be the continuous observation proc-
ess associated with the nonlinear process

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o$$

for all $t \in [t_o, t_f]$.
Let $\{y(t_i)\}$, $i = 1, 2, \cdots k$ be the sequence of obser-
vations at any k time points on $[t_o, t_f]$, such that $k \geq \frac{n}{m}$
and $y(t_i)$ has a strong derivative at x_o. Then the non-
linear system is locally observable at x_o under the con-
tinuous observation process $h(t, x(t))$; $t \in [t_o, t_f]$ with

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input u if

$$\sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial x_o} \right]^T \left[\frac{\partial y(t_i)}{\partial x_o} \right]$$

is nonsingular.

Application of the foregoing theorems require a minimum of $k = \frac{n}{m}$ observation times and, in general, an a priori knowledge of the solution so that an expression can be obtained for $y(t_i) = h(t_i, g(t_i, x_o, u))$. An alternative test for local observability, which avoids these problems, can be obtained by defining a recursion relationship. Admissibility conditions of the functions $f(\cdot,\cdot,\cdot)$, $h(\cdot,\cdot)$, and $u(\cdot)$ as defined in Section 2.1 will have to be made somewhat more restrictive. For the following theorems $f(\cdot,\cdot,\cdot)$ is continuous and has continuous partial derivatives with respect to all of its arguments up to and including order n - 1 and continuous mixed partial derivatives of the same order. Also let $u(\cdot)$ be continuous and have continuous derivatives of order n - 1. Let $h(\cdot,\cdot)$ be continuous and have continuous partial derivatives with respect to both of its arguments up to and including order n and continuous mixed partial derivatives of the same order. Now define the following recursion relationship:

$$y(t, x) \equiv F_o(t, x) = h(t, x(t))$$

$$\frac{dy}{dt} \equiv F_1(t, x, u) = \frac{\partial F_o}{\partial t} + \frac{\partial F_o}{\partial x} f(t, x, u)$$

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$$\frac{d^2 y}{dt^2} \equiv F_2(t, x, u, \dot{u}) = \frac{\partial F_1}{\partial t} + \left[\frac{\partial F_1}{\partial x}\right]^T f(t,x,u) + \left[\frac{\partial F_1}{\partial u}\right]^T \frac{du}{dt}$$

$$\vdots$$

$$\frac{d^n y}{dt^n} \equiv F_n(t,x,u,\dot{u}, \cdots \frac{du^{n-1}}{dt^{n-1}}) = \frac{\partial F_{n-1}}{\partial t}$$

$$+ \left[\frac{\partial F_{n-1}}{\partial x}\right]^T f(t,x,u) + \left[\frac{\partial F_{n-1}}{\partial u}\right]^T \frac{du}{dt}$$

$$+ \left[\frac{\partial F_{n-1}}{\partial \dot{u}}\right]^T \frac{d^2 u}{dt^2} + \cdots + \left[\frac{\partial F_{n-1}}{\partial u^{n-2}}\right]^T \frac{d^{n-1} u}{dt^{n-1}}$$

$$(2.1.14)$$

An nm - vector $\bar{F}(t,x)$ is formed as follows:

$$\bar{F}(t,x) = \left[F_o^T(t,x) F_1^T(t,x) \cdots F_{n-1}^T(t,x)\right]^T \qquad (2.1.15)$$

Theorem 2.6

Let Y be the discrete observation process formed by
the components $y(t_i)$, $i = 1, 2, \cdots k$ with $t_i \, \varepsilon \left[t_o, t_f\right]$
which is associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o$$

with the unique solution $g(t, x_o, u)$ for all $t \, \varepsilon \left[t_o, t_f\right]$.
Let $\bar{F}(t, x)$ be the nm - vector defined by Eq (2.1.15) with
components obtained from the recursion relation Eq (2.1.14).
If for one of the observation times (say $t^* \, \varepsilon \, \{t_i\}$), $\bar{F}(t, x)$
has a strong derivative at $x(t^*)$ and the rank of the
Jacobian $\partial \bar{F}/\partial x$ evaluated at t^* (denoted by $\bar{F}'(t^*, x(t^*))$) is

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equal to n, then the nonlinear system is locally observable under the discrete observation process Y with input u.

Proof: From the definition of $\bar{F}$, we know that $\bar{F} : R^1 \times R^n \to R^{nm}$. Also from Theorem 2.1, if $\bar{F}'(t^*, x(t^*))$ has a rank of n, then an inverse function exists which maps points in a neighborhood of $\bar{F}(t^*, x(t^*))$ to a neighborhood of $x(t^*)$; i.e., $\bar{F}(t, x)$ is locally one-to-one at $x(t^*)$. Now we also know from the uniqueness and continuity of the solutions $x(t) = g(t, x_o, u)$ that there is a one-to-one mapping from a set of initial conditions in an open neighborhood of x_o^* which yield solutions in the open neighborhood δ^* of $x(t^*)$. Thus for observations in a neighborhood of t^* we have established the local one-to-one mappings:
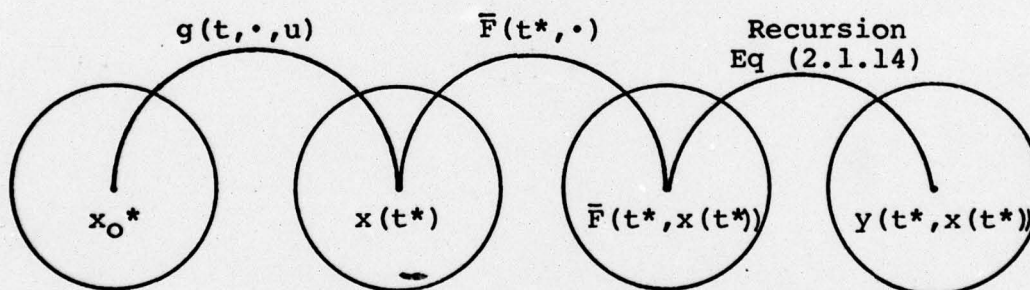
Given an observation $y(t^*, x(t^*))$, we use the recursion relation to obtain $\bar{F}(t^*, x(t^*))$ which has dimension nm and has a rank of n. Knowing $\bar{F}(t^*, x(t^*))$ uniquely determines $x(t^*)$ and $x(t^*)$ uniquely determines x_o^*.

Q.E.D.

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Note that this theorem relies on the system differential equation having a unique solution on $[t_o, t_f]$. Appendix B gives a brief summary of conditions which will assure a unique solution.

The previous theorem can be easily extended to a continuous observation process.

Theorem 2.7

Let $y(t) = h(t, x(t))$ be the continuous observation process associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o; \ t \ \varepsilon \ [t_o, t_f]$$

with a unique solution $g(t, x_o, u)$ for all $t \ \varepsilon \ [t_o, t_f]$. If there exists a $t^* \ \varepsilon \ [t_o, t_f]$ such that $\bar{F}(t, x)$ has a strong derivative at $x(t^*)$ and the rank of the Jacobian $\partial\bar{F}/\partial x$ evaluated at t^* is equal to n, then the nonlinear system is locally observable under the observation process $y(t)$, for all $t \ \varepsilon \ [t_o, t_f]$ with input u.

Proof: Combine Theorem 2.6 with the proof of Theorem 2.4. Consider the previous example where

$$\dot{x}_1(t) = x_1(t) \qquad\qquad x_1(0) = x_{o1}$$
$$\text{with}$$
$$\dot{x}_2(t) = 2x_2(t) \qquad\qquad x_2(0) = x_{o2}$$

for $t \ \varepsilon \ [0, t_f]$. The observations are

$$y(t_i) = x_1^2(t_i) + x_2^2(t_i) \qquad i = 1, 2, 3.$$

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Using the recursion relation Eq (2.1.14), we get

$$F_0 = y(t, x) = x_1^2(t) + x_2^2(t)$$

$$F_1 = \frac{dy}{dt} = 2x_1(t)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t)$$

$$= 2x_1^2(t) + 4x_2^2(t)$$

This yields

$$\bar{F}(t, x(t)) = \begin{bmatrix} x_1^2(t) + x_2^2(t) \\ 2x_1^2(t) + 4x_2^2(t) \end{bmatrix}$$

so that

$$\bar{F}'(t, x(t)) = \begin{bmatrix} 2x_1(t) & 2x_2(t) \\ 4x_1(t) & 8x_2(t) \end{bmatrix}$$

From the above, we see that the $\bar{F}'(t, x)$ has rank two for $x_1(t_o) \neq 0$ and $x_2(t_o) \neq 0$. Thus the system is locally identifiable for $x_o \in \{R^2 \sim \{x_{o1} = 0 \text{ or } x_{o2} = 0\}\}$.

2.3 Global Observability of Nonlinear Dynamical Systems

In the previous section we discussed local one-to-one correspondence and observability. Sufficient conditions were found to assure a local one-to-one correspondence between initial conditions in a neighborhood of x_o and the observation process. The question arises as to what conditions are necessary to assure that a one-to-one correspondence exists for any $x_o \in X_o \subset R^n$. It turns out that a

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local one-to-one correspondence at each point $x_o \, \epsilon \, X_o$ is not sufficient. A counter-example to such a proposition is obtained from the previous example. That is,

$$\dot{x}_1(t) = x_1(t)$$
$$\dot{x}_2(t) = 2x_2(t) \qquad t \, \epsilon \, [0, \, t_f] \qquad (2.3.1)$$

and

$$y(t_i) = x_1^2(t_i) + x_2^2(t_i) \qquad i = 1, \, 2, \, 3 \qquad (2.3.2)$$

We found the system to be locally observable for $x_o \, \epsilon \, \{R^2 \sim \{x_{o1} = 0 \text{ or } x_{o2} = 0\}\}$. Recall that the solution has the form

$$x_1(t) = x_{o1} \, e^t$$
$$x_2(t) = x_{o2} \, e^{2t} \qquad (2.3.3)$$

It is clear that solutions with initial conditions

$$x_o^1 = \begin{bmatrix} x_{o1} \\ x_{o2} \end{bmatrix} \quad \text{and} \quad x_o^2 = \begin{bmatrix} -x_{o1} \\ x_{o2} \end{bmatrix}$$

will result in identical observations; i.e., $Y(x_o^1) = Y(x_o^2)$.

This exemplifies the need for additional theory to establish conditions for global observability.

<u>Definition</u>: The mapping $F : D \, CR^n \rightarrow R^n$ is <u>norm-coercive</u> on an open set $D_o \, C \, D$ if, for any $\gamma > 0$, there is a closed,

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bounded set $D_\gamma \subset D_o$ such that $||F(x)|| > \gamma$ for all
$x \in D_o \sim D_\gamma$.

Note that if $D_o = R^n$, then F is norm-coercive if and
only if $\lim_{||x|| \to \infty} ||F(x)|| = \infty$. It is apparent that if
$\lim_{||x|| \to \infty} ||F(x)|| = \infty$, then for a $\gamma > 0$ we can choose a D_γ such
that for $x \in R^n \sim D_\gamma$, $||x||$ will be sufficiently large so
that $||F(x)|| > \gamma$. On the other hand, if F is norm-
coercive, then for $\gamma \to \infty$, it is necessary that
$\lim_{||x|| \to \infty} ||F(x)|| = \infty$ to ensure that $||F(x)|| > \gamma$ for all
$x \in R^n \sim D_\gamma$.

Definition: The set D is <u>path-connected</u> if, for any two
points x, y $\in$ D, there is a continuous mapping $p : [0,1] \to D$
such that $p(0) = x$ and $p(1) = y$.

Theorem 2.8

Let DCR^n be open and path-connected and assume that
$F : DCR^n \to R^n$ is a local homeomorphism at each point of D.
Then F is a global homeomorphism of D onto R^n if and only
if F is norm-coercive on D.

Proof: See Ref 39:137.

Theorem 2.8 is developed based on the continuation
property which is a relatively standard tool in the theory
of ordinary differential equations.

Definition: The mapping $F : DCR^n \rightarrow R^n$ has the continuation property for a given continuous function $q : [0,1] \subset R^1 \rightarrow R^n$ if the existence of a continuous function $p : [0,a) \rightarrow D$, $a \in (0,1]$ such that $F(p(t)) = q(t)$ for all $t \in [0,a)$ implies that $\lim_{t \rightarrow a^-} p(t) = p(a)$ exists with $p(a) \in D$, and $F(p(a))=q(a)$.

A discussion of this property is contained in Ref 39:133. In all of the theorems where norm-coerciveness is stated, one can replace "norm-coercive" by "has the continuation property" and the theorem will remain valid. Norm-coerciveness is stated in the theorems since, in most cases, the norm-coercive condition is probably easier to verify than the continuation property. It can be shown that the continuation property holds for all continuously differentiable functions (Ref 39:140). Recall from above that "continuously differentiable" means a continuous Gateaux-derivative (and thus a continuous Frechet derivative). Combining this with Theorem 2.8 results in the following useful theorem.

Theorem 2.9

Let D be open and path-connected and assume that $F : DCR^n \rightarrow R^n$ is continuously differentiable on D. If the Jacobian matrix $F'(x)$ is nonsingular for all $x \in D$, then F is one-to-one on D.

With the above definitions and theorems as background, we can now make statements similar to Theorems 2.8 and 2.9 for a mapping $g : X \subset R^n \rightarrow Y \subset R^k$ where $k \geq n$.

Theorem 2.10

Suppose $g : X \subset R^n \to R^k$ where $k \geq n$. Let $X \subset X_{pc}$ where $X_{pc} \subset R^n$ is open and path-connected. Assume that for all $x \in X_{pc}$ g has a strong derivative at x and we can find a common $n \times n$ submatrix of dg/dx which is nonsingular. (By common submatrix, we mean it is always composed of the same rows of dg/dx). Define a mapping $g_1 : X_{pc} \subset R^n \to R^n$ where $g_1(x)$ is composed of the n components of $g(x)$ which correspond to the nonsingular $n \times n$ submatrix of dg/dx. Then g is a one-to-one mapping of X onto R^k if g_1 is norm-coercive on X_{pc}.

Proof: If $k = n$, then we let $X = X_{pc}$ and $g = g_1$ so that this becomes a statement of Theorem 2.8.

For $k > n$, suppose g_1 is norm-coercive on X_{pc}. Then from Theorem 2.6 we know g_1 is a homeomorphism of X_{pc} onto R^n. Now suppose g is not a one-to-one mapping of X onto R^k. Then there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $g(x_1) = g(x_2)$. But this contradicts g_1 being a homeomorphism since from the definition of g_1, this would require $g_1(x_1) = g_1(x_2)$.

Q.E.D.

Note that if $X = R^n$, the norm-coerciveness condition can be replaced by $\lim_{\|x\| \to \infty} \|g_1(x)\| = \infty$ in Theorem 2.10.

Corollary: Suppose $g : X \subset R^n \to R^k$ where $k \geq n$ and X is open and path-connected. If there exists a g_1 formed from

n components of g such that g_1 is continuously differentiable on X and $g_1'(x)$ is nonsingular for all $x \in X$, then g is a one-to-one mapping on X.

Proof: Follows directly from Theorems 2.9 and 2.10.

Theorem 2.10 and its corollary allow us to state sufficient conditions for a nonlinear process to be <u>globally</u> <u>observable</u> <u>under</u> <u>the</u> <u>observation</u> <u>process</u> <u>Y</u> <u>for</u> <u>all</u> $\underline{x_o} \in \underline{X_o}$ <u>with</u> <u>input</u> <u>u</u>. We can make such a statement when there is a one-to-one correspondence between any $x_o \in X_o \subset R^n$ and the observation vector Y for an input u. Refer to Section 2.1 for the definitions of the nonlinear process and admissibility conditions on $f(\cdot,\cdot,\cdot)$, $u(\cdot)$ and $h(\cdot,\cdot)$. Recall that the dimension of Y is mk. Also, $mk \geq n$ where n is the dimension of $x(t)$.

<u>Theorem 2.11</u>

Let Y be the mk observation vector associated with the nonlinear process described by

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o$$

Suppose that for all $x_o \in X_{pc}$ where $X_{pc} \subset R^n$ is open and path-connected, Y has a strong derivative at x_o and there is a common $n \times n$ submatrix of $\partial Y / \partial x_o$ which is nonsingular. Define a mapping $Y_1 : X_{pc} \to R^n$ such that $Y_1(x_o)$ is equal to the n components of $Y(x_o)$ which yield the nonsingular $n \times n$ submatrix of $\partial Y / \partial x_o$. Then there is a one-to-one

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correspondence between Y and any initial state vector $x_o \in X_o \subset X_{pc}$ if (1) Y_1 is norm-coercive on X_{pc} <u>or</u> (2) Y_1 is continuously differentiable on X_{pc}.

Proof: This follows directly from Theorem 2.10 and its corollary.

Note that if $X_{pc} = R^n$ in Theorem 2.11, then the norm-coercive condition is replaced by

$$\lim_{||x_o|| \to \infty} ||Y_1(x_o)|| = \infty$$

Theorem 2.11 gives sufficient conditions for a nonlinear process to be globally observable. For some problems, a more practical set of conditions can be obtained from the properties of monotone functions.

<u>Definition</u>: A mapping $F : D \subset R^n \to R^n$ is <u>monotone</u> on $D_o \subset D$ if

$$[F(x) - F(y)]^T [x - y] \geq 0$$

for all $x, y \in D_o$. F is <u>strictly</u> <u>monotone</u> on D_o if strict inequality holds whenever $x \neq y$, and <u>uniformly</u> <u>monotone</u> if there is a $\gamma > 0$ so that

$$[F(x) - F(y)]^T [x - y] \geq \gamma (x - y)^T (x - y)$$

for all $x, y \in D_o$.

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Theorem 2.12

Let $F : D \subset R^n \rightarrow R^n$ be strictly monotone on $D_o \subset D$. Then F is a one-to-one mapping on D_o.

Proof: From the definition of strict monotonicity

$$\left[F(x) - F(y) \right]^T \left[x - y \right] > 0$$

for all $x, y \in D_o$ $x \neq y$.

If F were not one-to-one, then there exists an $x, y \in D_o$ such that $F(x) = F(y)$ with $x \neq y$. This contradicts the defining inequality.

<div align="right">Q.E.D.</div>

Theorem 2.13

Suppose $g : X \subset R^n \rightarrow R^k$ where $k \geq n$. If there exists a g_1 formed from n components of g such that g_1 is strictly monotone on X, then g is one-to-one on X.

Proof: Theorem 2.12 implies g_1 is one-to-one on X. Following the same procedure as the proof of Theorem 2.10 we can show that this requires g to be one-to-one on X.

<div align="right">Q.E.D.</div>

Theorem 2.14

Let Y be the mk observation vector associated with the nonlinear process described by

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \in X_o$$

so that $Y : R^n \rightarrow R^{mk}$. If there exists a Y_1 formed from n

components of Y such that Y_1 is strictly monotone on $X_o \subset R^n$, then there is a one-to-one correspondence between Y and any initial state vector $x_o \varepsilon X_o \subset R^n$.

Proof: Follows directly from Theorem 2.13.

Another set of conditions, which may be of use in some cases, can be derived. First a couple of introductory theorems will be established.

Definition: A subset $D_o \subset R^n$ is <u>convex</u> if, whenever it contains x and y, it also contains $\lambda x + (1 - \lambda)y$ for all $x, 0 \leq \lambda \leq 1$. Note that the set $\{z : z = \lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1\}$ is called the line segment joining x and y.

Theorem 2.15

Let $F : D \subset R^n \rightarrow R^n$ be continuously differentiable on an open convex set $D_o \subset D$. If dF/dx is positive definite for all $x \varepsilon D_o$, then F is strictly monotone on D_o.

Proof: See Ref 39:142.

Corollary: If $F : D \subset R^n \rightarrow R^n$ is continuously differentiable on an open convex set $D_o \subset D$ and dF/dx is positive definite for all $x \varepsilon D_o$, then F is one-to-one on D_o.

Proof: From Theorem 2.15, F is strictly monotone on D_o. From Theorem 2.12, this is sufficient to assure F is one-to-one.

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Following the same approach that was used in arriving at Theorem 2.14 from Theorem 2.12, we obtain

Theorem 2.16

Let Y be the mk observation vector associated with the nonlinear process

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \; \varepsilon \; X_o$$

so that $Y : X_o \subset R^n \rightarrow R^{mk}$. Define $Y_1 : X_o \subset R^n \rightarrow R^n$ by taking any n components of $Y(x_o)$. Then there is a one-to-one correspondence between Y and any initial state vector $x_o \; \varepsilon \; D_o \subset X_o$ (i.e., the nonlinear system is globally observable on D_o under the observation process Y for input u) if Y_1 is continuously differentiable on an open, convex set $D_o \subset X_o$ and $\partial Y_1 / \partial x_o$ is positive definite for all $x_o \; \varepsilon \; D_o$.

Theorems 2.11, 2.14, and 2.16 can be used to make statements of sufficient conditions for global observability of nonlinear systems under continuous observations.

Theorem 2.17

Let $y(t) = h(t, x(t))$ be the continuous observation process associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \; \varepsilon \; X_o$$

for all $t \; \varepsilon \; [t_o, t_f]$.

Let Y be the discrete observation process vector formed by the components $y(t_i)$, i = 1, 2, $\cdot$ $\cdot$ $\cdot$ k with $k \geq \frac{n}{m}$ and t_i

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any k time points on $[t_o, t_f]$. Then the nonlinear process is globally observable on $X_o \subset R^n$ under the continuous observation $y(t)$ for all $t \in [t_o, t_f]$ with input u, if one of the following conditions is met:

(1) X_o is open and path-connected and there exists a Y_1 formed from n components of Y such that Y_1 has a strong derivative at x_o and $\partial Y_1/\partial x$ is nonsingular for all $x_o \in X_o$ and Y_1 is norm-coercive on X_o.

(2) X_o is open and path-connected and there exists a Y_1 formed from n components of Y such that Y_1 is continuously differentiable on X_o and $\partial Y_1/\partial x_o$ is nonsingular for all $x_o \in X_o$.

(3) There exists a Y_1 formed from n components of Y such that Y_1 is a strictly monotone function of x_o for all $x_o \in X_o$.

(4) X_o is an open, convex set and there exists a Y_1 formed from n components of Y such that Y_1 is continuously differentiable and $\partial Y_1/\partial x_o$ is positive definite for all $x_o \in X_o$.

Proof: Follows directly from Theorems 2.11, 2.14, and 2.16 using the same proof as used for Theorem 2.4.

Using the recursion relation of Eq (2.1.14), we can formulate another set of sufficient conditions for global observability.

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Theorem 2.18

Let Y be the discrete observation process formed by the components $y(t_i)$, $i = 1, 2, \cdots k$ which is associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \; \varepsilon \; X_o$$

with a unique solution $g(t, x_o, u)$ for all $t \; \varepsilon [t_o, t_f]$.

Let $\bar{F}(t, x)$ be the nm-vector defined by Eq (2.1.15). Let t^* denote any one of the observation times $\{t_i\}$ and let the solution $g(t, x_o, u)$ map the initial conditions $x_o \; \varepsilon \; X_o$ onto X^* at time t^*. Then the nonlinear process is globally observable on $X_o \subset R^n$ under the observation process Y with input u if one of the following conditions is satisfied:

(1) X^* is open and path-connected and there exists an $\bar{F}_1(t, x)$ formed from n components of $\bar{F}$ such that $\bar{F}_1'(t^*, x(t^*))$ is strong and is nonsingular for all $x(t^*) \; \varepsilon \; X^*$ and $\bar{F}_1(t^*, x(t^*))$ is norm-coercive on X^*.

(2) X^* is open and path-connected and there exists an $\bar{F}_1(t, x)$ formed from n components of $\bar{F}(t, x)$ which at t^* is continuously differentiable on X^* and $F_1'(t^*, x(t^*))$ is nonsingular for all $x(t^*) \; \varepsilon \; X^*$.

(3)   There exists an $\bar{F}_1(t, x)$ formed from n components of $\bar{F}(t, x)$ such that $\bar{F}(t^*, x(t^*))$ is a strictly monotone function of $x(t^*)$ for all $x(t^*) \; \varepsilon \; X^*$.

(4) X^* is an open, convex set and there exists an $\bar{F}_1(t, x)$ formed from n components of $\bar{F}(t, x)$ such that

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$\bar{F}_1(t^*, x(t^*))$ is continuously differentiable and
$\bar{F}_1'(t^*, x(t^*))$ is positive definite for all $x(t^*) \epsilon X^*$.

Proof: Combine Theorems 2.11, 2.14, and 2.16 with the proof of Theorem 2.6.

Theorem 2.18 can be extended to a continuous observation process very easily.

Theorem 2.19

Let $y(t) = h(t, x(t))$ for all $t \epsilon [t_o, t_f]$ be the continuous observation process associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \epsilon X_o$$

for all $t \epsilon [t_o, t_f]$.

Let $t^* \epsilon [t_o, t_f]$ and $g(t^*, x_o, u)$ map the initial conditions $x_o \epsilon X_o$ onto X^* at t^*. Then the nonlinear process is globally observable on $X_o C R^n$ under the continuous observation process $y(t)$ with input u if any one of the conditions of Theorem 2.18 is satisfied.

Proof: Combine Theorem 2.18 with the proof of Theorem 2.4.

Consider the example presented at the beginning of this section.

$$\dot{x}_1(t) = x_1(t)$$

$$\dot{x}_2(t) = 2x_2(t) \qquad\qquad t \epsilon [t_o, t_f]$$

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and

$$y(t_i) = x^2(t_i) + x^2(t_i) \qquad i = 1, 2, 3$$

so that the observation process is

$$Y = \begin{bmatrix} x_1^2(t_1) + x_2^2(t_1) \\ x_1^2(t_2) + x_2^2(t_2) \\ x_1^2(t_3) + x_2^2(t_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_{o1}^2 e^{2t_1} + x_{o2}^2 e^{4t_1} \\ x_{o2}^2 e^{2t_2} + x_{o2}^2 e^{4t_2} \\ x_{o1}^2 e^{2t_3} + x_{o2}^2 e^{4t_3} \end{bmatrix}$$

We showed that the nonlinear system was locally observable under the observation process on $X_o = \{R^2 \sim \{x_{o1} = 0$ or $x_{o2} = 0\}\}$; however, the system is not globally observable on X_o. Suppose we test for global observability on the set $X_o' \subset R^2$ where $X_o' = \{x_o \ni x_{o1} > 0, x_{o2} > 0\}$. That is X_o' is the following subset of R^2:

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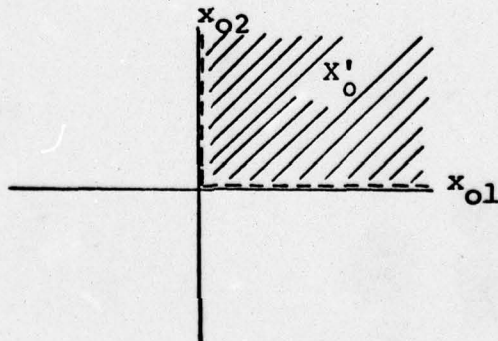
X_o' is open and path-connected. Let Y_1 be formed from the first two components of Y so that

$$Y_1 = \begin{bmatrix} x_{o1}^2\, e^{2t_1} + x_{o2}^2\, e^{4t_1} \\ x_{o1}^2\, e^{2t_2} + x_{o2}^2\, e^{4t_2} \end{bmatrix}$$

so that

$$\frac{dY_1}{dx_o} = \begin{bmatrix} 2x_{o1}\, e^{2t_1} & 2x_{o2}\, e^{4t_1} \\ 2x_{o1}\, e^{2t_2} & 2x_{o2}\, e^{4t_2} \end{bmatrix}$$

$\dfrac{dY_1}{dx_o}$ is nonsingular on X_o', so that by Theorem 2.11, the system is globally observable if Y_1 is norm-coercive on X_o'. If Y_1 is norm-coercive, then for any $\gamma > 0$, there must be a closed bounded set $D_\gamma \subset X_o'$ such that $||Y_1(x_o)|| > \gamma$ for $x_o \in X' \sim D_\gamma$. From Y_1 we see that $||Y_1|| > x_{o1}^2\, e^{2t_1}$ for $x_o \in X_o$, so that

$$x_{o1}^2\, e^{2t_1} > \gamma \implies ||Y_1|| > \gamma$$

Therefore we want to find an $x_{o1}(\gamma)$ such that

$$x_{o1}^2\, e^{2t_1} > \gamma$$

or

$$x_{o1} > \sqrt{\frac{\gamma}{e^{2t_1}}}$$

From this we see that given a $\gamma > 0$, we choose

$$D_\gamma = \{X_o' \sim \{x_{o1} > \sqrt{\frac{\gamma}{e^{2t_1}}} , \ x_{o2} > 0\}\}$$

This shows Y_1 is norm-coercive on X_o' and therefore the system is globally observable on X_o'. This illustrates that for nonlinear processes, there may be observability on only parts of R^n. Also, although not shown by the example, it is clear that observability of a nonlinear system depends on the input function u.

2.4 Observability of Linear Time Invariant Systems

Consider systems which are described by

$$\dot{x}(t) = A \ x(t) + B \ u(t) \qquad t \ \varepsilon \left[t_o, \ t_f \right], \ x(t_o) = x_o$$
$$(2.4.1)$$

and an observation process

$$y(t_i) = C \ x(t_i) + D \ u(t_i) \qquad i = 1, \ 2, \ \cdot \ \cdot \ \cdot \ k \qquad (2.4.2)$$

Suppose we apply Theorem 2.6 to this problem to test for observability. From the recursion relation Eq (2.1.14):

$$F_o = y(t) = C \ x(t) + D \ u(t)$$

$$F_1 = C \ \dot{x}(t) + D \ \dot{u}(t)$$
$$= C \ A \ x(t) + C \ B \ u(t) + D \ \dot{u}(t)$$

$$F_2 = C \ A \ \dot{x}(t) + C \ B \ \dot{u}(t) + D \ \ddot{u}(t)$$
$$= C \ A^2 \ x(t) + C \ AB \ u(t) + C \ B \ \dot{u}(t) + D \ \ddot{u}(t)$$

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$$F_3 = C A^2 \dot{x}(t) + C\, AB\, \dot{u}(t) + C\, B\, \ddot{u}(t) + D\, \dddot{u}(t)$$

$$= C A^3 x(t) + C A^2 B\, u(t) + C\, AB\, \dot{u}(t) + C\, B\, \ddot{u}(t) + D\, \dddot{u}(t)$$

.
.
.

$$F_{n-1} = C A^{n-1} x(t) + C A^{n-2} B\, u(t) + C A^{n-3} B\, \dot{u}(t) \cdots$$

$$+ C\, B\, \frac{\partial^{n-2} {}_2}{\partial t^{n-2}} + D\, \frac{\partial_1 {}^{n-1} u}{\partial t^{n-1}}$$

Also recall that

$$\bar{F} = \begin{bmatrix} F_o{}^T & F_1{}^T & \cdots & F_{n-1}{}^T \end{bmatrix}^T$$

and therefore

$$\frac{\partial \bar{F}}{\partial x} = \begin{bmatrix} \dfrac{\partial F_o}{\partial x} & \vdots & \dfrac{\partial F_1}{\partial x} & \vdots & \cdots & \vdots & \dfrac{\partial F_{n-1}}{\partial x} \end{bmatrix}^T$$

$$\frac{\partial \bar{F}}{\partial x} = \begin{bmatrix} C^T & \vdots & A^T C^T & \vdots & A^{T^2} C^T & \vdots & \cdots & \vdots & A^{T^{n-1}} C^T \end{bmatrix}^T \qquad (2.4.3)$$

Thus from Theorem 2.6, the system is locally observable if $\partial \bar{F}/\partial x$ has rank n.

To test for global observability we can use Theorem 2.18. In addition to the condition $\partial \bar{F}/\partial x$ being of rank n, we must show that any set of n components of $\bar{F}$ is norm-coercive on the solution set X. For this case, the observation process is a mapping $Y(x_o) : x_o \in R^n \to R^n$. Recall from page 26 that for this case $\bar{F}$ is norm-coercive on R^n if

$$\lim_{\|x(t)\| \to \infty} \|\bar{F}(t,\, x(t))\| = \infty$$

This is seen to be true by looking at the components F_0, F_1, $\cdots$ F_{n-1} on the previous page. From the above we see that linear time invariant systems are globally observable on R^n if

$$\left[C^T \; \middle| \; A^T C^T \; \middle| \; A^{T^2} C^T \; \middle| \; \cdots \; \middle| \; A^{T^{n-1}} C^T \right]$$

is of rank n. This result agrees with that obtained from linear system theory. Note that observability does not depend on the input function u.

Clearly these same results apply for a continuous observation process.

2.5 Summary

In this chapter sufficient conditions have been established for determining local and global observability of nonlinear systems.

Local Observability. Theorems 2.2 and 2.3 state sufficient conditions for local observability of nonlinear systems under discrete time measurements. Theorem 2.6 provides alternative conditions based on a recursion relationship which can avoid the requirement of an a priori knowledge of the solution, but requires the system differential equation have a unique solution on $[t_0, t_f]$. Corresponding sufficient conditions for establishing local observability under continuous time measurements are stated in Theorems 2.4, 2.5, and 2.7.

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Global Observability. Theorem 2.11 extends the results of Theorem 2.3 for discrete time measurements to sufficient conditions for global observability based on a norm-coercive or a continuously differentiable condition. Theorems 2.14 and 2.16 state sufficient conditions for global observability under discrete time measurements using the properties of strictly monotone and convex functions respectively. Theorem 2.18 extends Theorem 2.6 to sufficient conditions for global observability under discrete measurements based on a recursion relation by using the results of Theorems 2.11, 2.14, and 2.16. Finally Theorems 2.17 and 2.19 extend the above results to sufficient conditions for global observability under continuous time measurements.

Chapter III

IDENTIFIABILITY OF NONLINEAR DYNAMICAL SYSTEMS

The previous chapter developed new results concerning observability of nonlinear systems under discrete and continuous observations. We now want to apply these results to the problem of identifying parameters of nonlinear dynamical systems.

3.1 Nonlinear State Equations and Definition of Identifiability

Consider the nonlinear systems described by

$$\dot{x}(t) = f(t,\ x(t),\ u(t),\ \phi) \qquad x(t_o) = x_o \qquad (3.1.1)$$

and the discrete observation process

$$y(t_i) = h(t_i,\ x(t_i),\ \phi) \qquad i = 1, 2,\ \cdot\ \cdot\ \cdot\ ,\ k$$
$$(3.1.2)$$

where $x(t)$ is an n vector, $y(t_i)$ is an m vector, u is an r vector describing the input function, ϕ is an s vector of constant parameters, and $t_i \in [t_o,\ t_f]$ for all i. Assume $f(\cdot,\cdot,\cdot,\cdot)$ has continuous partial derivatives with respect to t, x, and ϕ. u is a continuous function on $[t_o,\ t_f]$ and $h(\cdot,\cdot,\cdot)$ has continuous partial derivatives with respect to its three arguments.

A solution of Eq (3.1.1) is denoted by $g(t,\ x_o,\ u,\ \phi)$. In the identifiability problem, we are interested in finding sufficient conditions to guarantee a one-to-one correspondence

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between the parameter vector ϕ and the observed sequence $\{y(t_i)\}$. That is, we want to assure that the value of ϕ is uniquely determined from the knowledge of $u(t)$ and the output sequence $\{y(t_i)\}$ for all $t \in [t_o, t_f]$. As in Chapter II, the observation process is viewed as a single mk vector

$$Y = [y^T(t_1)y^T(t_2) \cdots y^T(t_k)]^T \qquad (3.1.3)$$

The parameter vector ϕ of the nonlinear system is <u>identifiable</u> under the observation process Y with input u if there is a one-to-one correspondence between the parameter vector ϕ and the observation vector $Y(\phi)$. The parameter vector ϕ is <u>locally identifiable</u> at ϕ when the one-to-one correspondence holds in a neighborhood of ϕ. The parameter vector is <u>globally identifiable</u> on Φ_o under the observation process Y when there is a one-to-one correspondence between ϕ and Y for all $\phi \in \Phi_o \subset R^s$ with the input u. By defining an augmented state vector

$$x_a(t) = \begin{bmatrix} x(t) \\ \phi \end{bmatrix} \qquad (3.1.4)$$

it becomes clear that identification of ϕ is a special case of the observability problem. That is

$$\dot{x}_a(t) = \begin{bmatrix} f(t, x(t), u(t), \phi) \\ 0 \end{bmatrix} \qquad (3.1.5)$$

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with

$$x_a(t_o) = \begin{bmatrix} x(t_o) \\ \phi \end{bmatrix} \qquad (3.1.6)$$

Thus identifying ϕ becomes part of the observability problem for $x_a(t)$. In this chapter, we will assume the initial conditions $x(t_o)$ are known and use the results of Chapter II to state conditions for identifiability.

3.2 Local Identifiability of Nonlinear Dynamical Systems

In this section, we will follow the same approach as Section 2.2 in deriving sufficient conditions for the parameter vector ϕ of a nonlinear system to be locally identifiable.

Theorem 3.1

Let Y be the mk observation vector associated with the nonlinear process

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o$$

with the dimension of Y equal to or greater than the parameter vector ϕ; i.e., mk $\geq$ s and let Y have a strong derivative at x_o. Then the parameter vector ϕ is locally observable under the observation process Y with input u and initial conditions x_o if $\partial Y/\partial\phi$ has rank s.

Proof: This follows directly from Theorem 2.1.

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Theorem 3.2

Let Y have a strong derivative at x_O. Then the parameter ϕ is locally identifiable under the observation process Y with input u and initial conditions x_O if

$$\sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial \phi} \right]^T \left[\frac{\partial y(t_i)}{\partial \phi} \right]$$

is nonsingular.

Proof: This is a special case of Theorem 2.3.

These theorems can be easily extended to continuous observations as follows:

Theorem 3.3

Let $y(t) = h(t, x(t), \phi)$ for all $t \in \left[t_O, t_f \right]$ be the continuous observation process associated with the nonlinear process

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_O) = x_O$$

Let Y be the discrete observation vector formed from $\{y(t_i)\}$ $i = 1, 2, \cdots, k$, the sequence of observations at any k time points on $\left[t_O, t_f \right]$ such that $k \geq \frac{n}{m}$ and let Y have a strong derivative at x_O. Then the parameter ϕ is locally identifiable under the continuous observation process $y(\cdot)$ for all $t \in \left[t_O, t_f \right]$ with input u and initial conditions x_O

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if either of the following conditions is met:

(1) $\partial Y/\partial \phi$ has rank s.

(2) $\sum_{i=1}^{k} \left[\dfrac{\partial y(t_i)}{\partial \phi}\right]^T \left[\dfrac{\partial y(t_i)}{\partial \phi}\right]$ is nonsingular.

Proof: This follows from Theorems 2.4 and 2.5.

The recursion relation of Eq (2.1.14) can be used to derive additional sufficient conditions for local identifiability. The recursion can be used to derive the functions F_0, F_1, $F_2 \cdots$, F_{n+s-1}. Then an $(n + s)m$ vector is formed:
$$\bar{F}(t, \phi) = \left[F_0^T(t, \phi)F_1^T(t, \phi) \cdots F_{n+s-1}^T(t, \phi)\right].$$ Following the same approach as Theorem 2.6, we can state the following theorem.

Theorem 3.4

Let Y be the discrete observation process formed by the components $y(t_i)$, $i = 1, 2, \cdots k$ with $t_i \in [t_o, t_f]$ which is associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o$$

with a unique solution $g(t, x_o, u, \phi)$ for all $t \in [t_o, t_f]$. Let $\bar{F}(t, \phi)$ be the $(n + s)m$ vector formed from the recursion relation Eq (2.1.14). If for one of the observation times (say $t^* \in \{t_i\}$), $\bar{F}(t^*, \phi)$ has a strong derivative at x_o and the rank of the Jacobian $\partial \bar{F}/\partial \phi$ evaluated at t^* is equal to s, then the nonlinear system is locally identifiable under the discrete observation process Y with input u and initial

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conditions x_o.

Proof: Follows from Theorem 2.6.

A similar statement can be made for continuous observations.

Theorem 3.5

Let $\bar{F}(t, \phi)$ have a strong derivative at x_o. If for a continuous observation process, the rank of the Jacobian $\partial\bar{F}/\partial\phi$ evaluated at $t^* \varepsilon \left[t_o, t_f\right]$ is equal to s, then the nonlinear system is locally identifiable under the continuous observation process $y(\cdot)$ for all $t \varepsilon \left[t_o, t_f\right]$ with input u and initial condition x_o.

3.3 Global Identifiability of Nonlinear Dynamical Systems

As with the observability problem, local identifiability at each point $\phi \varepsilon \Phi$ is not sufficient to assure that a one-to-one correspondence exists between ϕ and Y for any $\phi \varepsilon \Phi \subset R^s$. We can draw upon the theorems in Chapter II concerning global observability to state some theorems on global identifiability.

Theorem 3.6

Let Y be the mk observation vector associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o$$

The parameter vector ϕ is globally identifiable on Φ under the observation process Y with input u and initial

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conditions x_o if one of the following conditions is satisfied:

(1) Φ is open and path-connected and there exists a Y_1 formed from s components of Y such that Y_1 has a strong derivative at x_o and $\partial Y_1/\partial\phi$ is nonsingular for all $\phi \epsilon \Phi$ and Y_1 is norm-coercive on Φ.

(2) Φ is open and path-connected and there exists a Y_1 formed from s components of Y such that Y_1 is continuously differentiable and $\partial Y_1/\partial\phi$ is nonsingular for all $\phi \epsilon \Phi$.

(3) There exists a Y_1 formed from s components of Y such that Y_1 is a strictly monotone function of ϕ for all $\phi \epsilon \Phi$.

(4) Φ is an open, convex set and there exists a Y_1 formed from s components of Y such that Y_1 is continuously differentiable and $\partial Y_1/\partial\phi$ is positive definite for all $\phi \epsilon \Phi$.

Proof: Follows from Theorems 2.11, 2.14, and 2.16.

Following the same approach as for the observation problem, we can extend the above theorem to the continuous observation process $y(t) = h(t, x(t))$ for all $t \epsilon [t_o, t_f]$. Theorem 3.6 applies to this case by letting Y be formed from any k observations times of the continuous observation on $[t_o, t_f]$ with $k \geq \frac{s}{m}$.

The recursion relation of Eq (2.1.14) can be applied to the global identification process. As with the local identification, the recursion relation can be used to derive the functions F_o, F_1, F_2, $\cdots$ F_{n+s-1}. Then an $(n + s)m$ vector is formed as follows:

$$\bar{F}(t, \phi) = \left[F_o^T(t, \phi) F_1^T(t, \phi) \cdots F_{n+s-1}^T(t, \phi) \right]^T$$

$$(3.3.1)$$

Theorem 3.7

Let Y be the discrete observation process formed by the components $y(t_i)$, $i = 1, 2, \cdots , k$ with $t_i \in [t_o, t_f]$ which is associated with the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o$$

with a unique solution $g(t, x_o, u, \phi)$ for all $t \in [t_o, t_f]$. Let $\bar{F}(t, \phi)$ be the $(n + s)m$ vector formed from the recursion relation as indicated in Eq (3.3.1). Let t* denote any one of the observation times $\{t_i\}$ so that $\bar{F}(t^*, \phi)$ is the mapping $\bar{F} : \Phi \subset R^s \rightarrow R^{(n+s)m}$. Then the nonlinear process is globally identifiable on Φ under the observation process Y with input u and initial conditions x_o if one of the following conditions is satisfied:

(1) Φ is open and path-connected and there exists an $\bar{F}_1(t, \phi)$ formed from s components of $\bar{F}(t, \phi)$ such that $\partial\bar{F}/\partial\phi$ evaluated at t* is strong and is nonsingular for all $\phi \in \Phi$ and $\bar{F}_1(t^*, \phi)$ is norm-coercive on Φ.

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(2) Φ is open and path-connected and there exists an $\bar{F}_1(t, \phi)$ formed from s components of $\bar{F}(t, \phi)$ which at t* is continuously differentiable and $\partial\bar{F}_1/\partial\phi$ is nonsingular for all $\phi \in \Phi$.

(3) There exists an $\bar{F}_1(t, \phi)$ formed from s components of $\bar{F}(t, \phi)$ such that $\bar{F}(t*, \phi)$ is a strictly monotone function for all $\phi \in \Phi$.

(4) Φ is an open, convex set and there exists an $\bar{F}_1(t, \phi)$ formed from s components of $\bar{F}(t, \phi)$ such that $\bar{F}_1$ is continuously differentiable and $\partial\bar{F}_1/\partial\phi$ evaluated at t* is positive definite for all $\phi \in \Phi$.

Proof: Follows from Theorem 2.18.

For continuous observations, the following theorem can be stated.

Theorem 3.8

Let $y(t) = h(t, x(t), \phi)$ for all $t \in [t_o, t_f]$ be the continuous observation process associated with the non-linear system

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o$$

with a unique solution $g(t, x_o, u, \phi)$ for all $t \in [t_o, t_f]$. Let $t* \in [t_o, t_f]$ so that $\bar{F}(t*, \phi)$ is the mapping $\bar{F} : \Phi \subset R^s \rightarrow R^{(n+s)m}$. Then the nonlinear process is globally identifiable on Φ under the continuous observation

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process y(t) for all $t \in [t_o, t_f]$ with input u and initial conditions x_o if any one of the conditions of Theorem 3.9 is satisfied.

3.4 Identifiability of Linear Time Invariant Systems

Consider systems which are described for all $t \in [t_o, t_f]$

$$\dot{x}(t) = A(\phi)x(t) + B(\phi)u(t) \qquad x(t_o) = x_o \qquad (3.4.1)$$

with a discrete observation process

$$y(t_i) = C(\phi)x(t_i) + D(\phi)u(t_i) \qquad i = 1, 2, \cdots k \quad (3.4.2)$$

Suppose we use Theorem 3.4 to test for local identifiability of ϕ. From the recursion relation

$$F_0 = C(\phi)x(t) + D(\phi)u(t)$$

$$F_1 = C(\phi)\dot{x}(t) + D(\phi)\dot{u}(t)$$

$$= C(\phi)A(\phi)x(t) + C(\phi)B(\phi)u(t) + D(\phi)\dot{u}(t)$$

$$F_2 = C(\phi)A(\phi)\dot{x}(t) + C(\phi)B(\phi)\dot{u}(t) + D(\phi)\ddot{u}(t)$$

$$= C(\phi)A^2(\phi)x(t) + C(\phi)A(\phi)B(\phi)u(t) + C(\phi)B(\phi)\dot{u}(t)$$
$$+ D(\phi)\ddot{u}(t)$$

$$F_3 = C(\phi)A^3(\phi)x(t) + C(\phi)A^2(\phi)B(\phi)u(t) + C(\phi)A(\phi)B(\phi)\dot{u}(t)$$
$$+ C(\phi)B(\phi)\ddot{u}(t) + D(\phi)\dddot{u}(t)$$

$$\vdots$$

$$F_{n+s-1} = C(\phi)A^{n+s-1}(\phi)x(t) + C(\phi)A^{n+s-2}(\phi)B(\phi)u(t)$$
$$+ C(\phi)A^{n+s-3}B(\phi)u(t) \cdots + C(\phi)A(\phi)B(\phi)\frac{\partial^{n+s-3}u}{\partial t^{n+s-3}}$$
$$+ C(\phi)B(\phi)\frac{\partial^{n+s-2}u}{\partial t^{n+s-2}} + D(\phi)\frac{\partial^{n+s-1}u}{\partial t^{n+s-1}}$$

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From this we obtain

$$
\bar{F} = \begin{bmatrix}
C(\phi)x(t) + D(\phi)u(t) \\
C(\phi)A(\phi)x(t) + C(\phi)B(\phi)u(t) + D(\phi)\dot{u}(t) \\
C(\phi)A^2(\phi)x(t) + C(\phi)A(\phi)B(\phi)u(t) + C(\phi)B(\phi)u(t) \\
\qquad\qquad + D(\phi)\ddot{u}(t) \\
\vdots \\
C(\phi)A^{n+s-1}(\phi)x(t) + C(\phi)A^{n+s-2}(\phi)B(\phi)u(t)\cdots \\
\qquad\qquad + D(\phi)\dfrac{\partial^{n+s-1}u(t)}{\partial t^{n+s-1}}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
C(\phi) & D(\phi) & 0 & \cdots & 0 \\
C(\phi)A(\phi) & C(\phi)B(\phi) & D(\phi) & & 0 \\
C(\phi)A^2(\phi) & C(\phi)A(\phi)B(\phi) & C(\phi)B(\phi) & & \vdots \\
\vdots & \vdots & \vdots & & \vdots \\
C(\phi)A^{n+s-1}(\phi) & C(\phi)A^{n+s-2}(\phi)B(\phi) & C(\phi)A^{n+s-3}(\phi)B(\phi) & \cdots & D(\phi)
\end{bmatrix}
\cdot
$$

$$
\begin{bmatrix}
x(t) \\
u(t) \\
\dot{u}(t) \\
\vdots \\
\dfrac{\partial^{n+s-1}u}{\partial t^{n+s-1}}
\end{bmatrix}
$$

$$
\triangleq \begin{bmatrix} \bar{F}_1 & \vdots & \bar{F}_2 & \vdots & \bar{F}_3 & \vdots & \cdots & \vdots & \bar{F}_{n+s} \end{bmatrix}
\begin{bmatrix}
x(t) \\
u(t) \\
\vdots \\
\dfrac{\partial^{n+s-1}u(t)}{\partial t^{n+s-1}}
\end{bmatrix}
$$

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Note that the nonzero elements of $\bar{F}_3$ through $\bar{F}_{n+s}$ are contained in $\bar{F}_2$.

From above we have

$$\bar{F} = \bar{F}_1\, x(t) + \bar{F}_2\, x(t) + \cdots + \bar{F}_{n+s}\, \frac{\partial^{n+s-1} u(t)}{\partial t^{n+s-1}}$$

Thus the derivative of $\bar{F}$ with respect to the vector ϕ is

$$\frac{\partial \bar{F}}{\partial \phi} = \left[\frac{\partial \bar{F}_1}{\partial \phi_1}\, x(t) \;\middle|\; \frac{\partial \bar{F}_1}{\partial \phi_2}\, x(t) \;\middle|\; \cdots \;\middle|\; \frac{\partial \bar{F}_1}{\partial \phi_s}\, x(t) \right]$$

$$+ \left[\frac{\partial \bar{F}_2}{\partial \phi_1}\, u(t) \;\middle|\; \frac{\partial \bar{F}_2}{\partial \phi_2}\, u(t) \;\middle|\; \cdots \;\middle|\; \frac{\partial \bar{F}_2}{\partial \phi_2}\, u(t) \right] + \cdots$$

$$+ \left[\frac{\partial \bar{F}_{n+s}}{\partial \phi_1} \frac{\partial^{n+s-1} u(t)}{\partial t^{n+s-1}} \;\middle|\; \cdots \;\middle|\; \frac{\partial F_{n+s}}{\partial \phi_s} \frac{\partial^{n+s-1} u(t)}{\partial t^{n+s-1}} \right]$$

$$(3.4.3)$$

From Eq (3.4.3), it is seen that $\partial \bar{F}/\partial \phi$ will have rank s if either of the following conditions is satisfied:

(1) $\dfrac{\partial \bar{F}_1}{\partial \phi}$ has rank s

(2) $\dfrac{\partial \bar{F}_2}{\partial \phi}$ has rank s

where

$$\bar{F}_1 = \begin{bmatrix} C(\phi) \\ C(\phi)A(\phi) \\ C(\phi)A^2(\phi) \\ \vdots \\ C(\phi)A^{n+s-1}(\phi) \end{bmatrix}$$

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$$\bar{F}_2 = \begin{bmatrix} D(\phi) \\ C(\phi)B(\phi) \\ C(\phi)A(\phi)B(\phi) \\ \cdot \\ \cdot \\ \cdot \\ C(\phi)A^{n+s-2}(\phi)B(\phi) \end{bmatrix}$$

and

$$\frac{\partial \bar{F}_i}{\partial \phi} = \begin{bmatrix} \dfrac{\partial \bar{F}_i}{\partial \phi_1} & \Big| & \dfrac{\partial \bar{F}_i}{\partial \phi_2} & \Big| & \cdots & \Big| & \dfrac{\partial \bar{F}_i}{\partial \phi_s} \end{bmatrix}$$

If <u>any</u> of the above sufficient conditions are satisfied, then the parameter vector ϕ is locally identifiable. Note that the first condition is independent of input; i.e., no $B(\phi)$, $D(\phi)$, or $u(t)$ are required. Therefore condition one allows checking for local identifiability independent of input. The second condition is independent of the state $x(t)$, but requires a nonzero input $u(t)$. This second condition agrees with that derived recently by Grewal and Glover (Ref 12) using linear system theory. The second condition may be easier to apply than condition one in some cases, but clearly can only be applied where $B(\phi) \neq 0$. Note that these two conditions are analogous to separating a linear system solution into the zero input solution and zero initial-state solution, with the total solution as the sum.

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The above results apply to either discrete or continuous observations. To test for global identifiability, one would have to assure that $\bar{F}$ was norm-coercive or continuously differentiable on an open and path-connected Φ.

Consider the following examples:

Example 1

$$\dot{x}_1(t) = - (\phi_1 + \phi_2)x_1(t) + \phi_2 x_2(t) + u(t)$$

$$\dot{x}_2(t) = \phi_2 x_1(t) - \phi_3 x_2(t)$$

with observation process

$$y(t) = x_1(t)$$

This is linear time invariant system with

$$A(\phi) = \begin{bmatrix} -(\phi_1 + \phi_2) & \phi_2 \\ \phi_2 & -\phi_3 \end{bmatrix} \qquad B(\phi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C(\phi) = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D(\phi) = \begin{bmatrix} 0 \end{bmatrix}$$

Therefore we can use the results of Section 3.4. First we can check if the system is identifiable with zero input. From Section 3.4 we use the first sufficient condition:

$$\bar{F}_1 = \begin{bmatrix} C(\phi) \\ C(\phi)A(\phi) \\ C(\phi)A^2(\phi) \\ C(\phi)A^3(\phi) \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \\ -(\phi_1+\phi_2) & \phi_2 \\ \begin{array}{l}+\phi_1{}^2+2\phi_1\phi_2+2\phi_2{}^2\end{array} & -\phi_1\phi_2+\phi_2{}^2-\phi_2\phi_3 \\ \begin{array}{l}-\phi_1{}^3-3\phi_1{}^2\phi_2-5\phi_1\phi_2{}^2 \\ \quad -\phi_2{}^2\phi_3-\phi_2{}^3\end{array} & \begin{array}{l}2\phi_1{}^2\phi_2+\phi_1\phi_2\phi_3 \\ \quad -\phi_2{}^2\phi_3+\phi_2\phi_3{}^2\end{array} \end{array}\right]$$

$$\frac{\partial \bar{F}_1}{\partial \phi} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ +2\phi_1+2\phi_2 & -\phi_2 & 2\phi_1+4\phi_2 \\ \begin{array}{l}-3\phi_1{}^2-6\phi_1\phi_2-5\phi_2{}^2\end{array} & 4\phi_1\phi_2+\phi_2\phi_3 & \begin{array}{l}-3\phi_1{}^2-10\phi_1\phi_2 \\ \quad -2\phi_2\phi_3-3\phi_2{}^2\end{array} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\phi_2+2\phi_2-\phi_3 & 0 & -\phi_2 \\ 2\phi_1{}^2+\phi_1\phi_3-2\phi_2\phi_3+\phi_3{}^2 & -\phi_2{}^2 & \phi_1\phi_2-\phi_2{}^2+2\phi_2\phi_3 \end{bmatrix}^T$$

Clearly this has rank three for $\phi_2 \neq 0$. Thus the system is locally identifiable on $\{R^3 \sim \{\phi_2 = 0\}\}$. We can also check the second condition developed in Section 3.4. The second condition is to determine if $\partial\bar{F}_2/\partial\phi$ has rank s where

$$\bar{F}_2 = \begin{bmatrix} D(\phi) \\ C(\phi)B(\phi) \\ C(\phi)A(\phi)B(\phi) \\ C(\phi)A^2(\phi)B(\phi) \\ C(\phi)A^3(\phi)B(\phi) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -(\phi_1+\phi_2) \\ \phi_1{}^2+2\phi_1\phi_2+2\phi_2{}^2 \\ -\phi_1{}^3-3\phi_1{}^2\phi_2-5\phi_1\phi_2{}^2-\phi_2{}^2\phi_3-\phi_2{}^3 \end{bmatrix}$$

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$$\frac{\partial \bar{F}_2}{\partial \phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \\ 2\phi_1 + 2\phi_2 & 2\phi_1 + 4\phi_2 & 0 \\ -3\phi_1{}^2 - 6\phi_1\phi_2 - 5\phi_2{}^2 & -3\phi_1{}^2 - 10\phi_1\phi_2 - 2\phi_2\phi_3 - 3\phi_2{}^2 & -\phi_2{}^2 \end{bmatrix}$$

Again this has rank three for $\phi_2 \neq 0$. The above result agrees with that obtained in Ref 25.

Example 2

$$\dot{x}_1(t) = -\phi_2 x_2(t)$$

$$\dot{x}_2(t) = -\phi_1 x_1{}^3(t) - \phi_2 x_2(t) + u^2(t)$$

$$y(t) = x_1(t)$$

Using Theorem 3.4, we form $\bar{F}$.

$$F_o = x_1(t)$$

$$F_1 = -\phi_2 x_2(t)$$

$$F_2 = -\phi_2(-\phi_1 x_1{}^3(t) - \phi_2 x_2(t) + u^2(t))$$

$$F_3 = +\phi_2 \phi_1 3 x_1{}^2 \dot{x}_1(t) + \phi_2{}^2 \dot{x}_2(t) - 2\phi_2 u(t) \dot{u}(t)$$

$$= -3\phi_2{}^2 \phi_1 x_1{}^2(t) x_2(t) + \phi_2{}^2(-\phi_1 x_1{}^3(t) - \phi_2 x_2(t) + u^2(t))$$

$$-2\phi_2 u(t)\dot{u}(t)$$

$$= -3\phi_1\phi_2{}^2 x_1{}^2(t) x_2(t) - \phi_1\phi_2{}^2 x_1{}^3(t) - \phi_2{}^3 x_2(t) + \phi_2{}^2 u^2(t)$$

$$-2\phi_2 u(t)\dot{u}(t)$$

so that

$$\bar{F} = \begin{bmatrix} x_1(t) \\ -\phi_2 x_2(t) \\ \phi_1\phi_2 x_1^3(t) + \phi_2^2 x_2(t) - \phi_2 u^2(t) \\ -3\phi_1\phi_2^2 x_1^2(t)x_2(t) - \phi_1\phi_2^2 x_1^3(t) - \phi_2^3 x_2(t) + \phi_2^2 u^2(t) - 2\phi_2 u(t)\dot{u}(t) \end{bmatrix}$$

$$\frac{\partial \bar{F}}{\partial \phi} = \begin{bmatrix} 0 & 0 \\ 0 & -x_2(t) \\ \phi_2 x_1^3(t) & \phi_1 x_1^3(t) + 2\phi_2 x_2(t) - u^2(t) \\ -3\phi_2^2 x_1^2(t)x_2(t) - \phi_2^2 x_1^3(t) & -6\phi_1\phi_2 x_1^2(t)x_2(t) - 2\phi_1\phi_2 x_1^3(t) \\ & -3\phi_2^2 x_2(t) + 2\phi_2 u^2(t) - 2u(t)\dot{u}(t) \end{bmatrix}$$

This matrix has rank two for $\phi_2 \neq 0$. Therefore the parameter vector ϕ is locally identifiable for $\phi \in \Phi = \{R^2 \sim \{\phi_2 = 0\}\}$.

3.5 Stochastic Identifiability of Parameters of Nonlinear Dynamical Systems

In this section we are interested in extending the previous results to the case in which the observations are corrupted by additive noise. To do this we must first define identifiability in terms of output distinguishability in a stochastic sense.

The nonlinear system is modeled by

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o \qquad (3.5.1)$$

for all t. The discrete observation process is corrupted

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by a zero-mean white Gaussian sequence $\{v(t_i)\}$; i.e.,

$$z(t_i) = y(t_i) + v(t_i) \qquad i = 1, 2, \cdots k \qquad (3.5.2)$$

where

$$y(t_i) = h(t_i, x(t_i), \phi) \qquad (3.5.3)$$

The objective is to find a stochastic analog to the sufficient conditions for identifying ϕ that were found for the deterministic case. Tse and Anton (50) define stochastic identifiability in terms of consistency in probability. We will use this definition which is independent of the estimation method used for evaluating the parameters.

Definition: A sequence of estimates $\{\phi_i\}_{i = 1, 2, \cdots n \cdots}$ is said to be consistent in probability if for any δ, ε arbitrarily small, there exists an $N(\varepsilon, \delta)$ such that for $n > N(\varepsilon, \delta)$

$$\Pr\{||\phi_n - \phi_o|| > \delta\} < \varepsilon \qquad (3.5.4)$$

where ϕ_o is the true value of the parameter. In other words, the sequence of estimates converges in probability to the true value ϕ_o as $n \to \infty$.

Definition: The parameter ϕ_o is said to be identifiable if there exists a sequence of estimates $\{\phi_i\}$ which is consistent in probability.

Definition: If a sequence of observations $Z_k = \{z(t_i)\}$, $i = 1, 2, \cdots k$ is made, then the __maximum likelihood estimate of__ ϕ is given by

$$\hat{\phi} = \max_{\phi \varepsilon \Phi} p(Z_k \mid \phi) \qquad (3.5.5)$$

With successive applications of Bayes Rule, an alternate expression for $p(Z_k \mid \phi)$ can be found.

$$
\begin{aligned}
p(Z_k \mid \phi) &\overset{\Delta}{=} p(z(t_1), z(t_2), \cdots z(t_k) \mid \phi) \\
&= p(z(t_k) \mid Z_{k-1}, \phi) p(Z_{k-1} \mid \phi) \\
&= p(z(t_k) \mid Z_{k-1}, \phi) p(z(t_{k-1}) \mid Z_{k-2}, \phi) p(Z_{k-2} \mid \phi) \\
&\quad \vdots \\
&= \prod_{j=1}^{k} p(z(t_j) \mid Z_{j-1}, \phi)
\end{aligned}
$$

$$(3.5.6)$$

The consistency of the maximum likelihood estimator has been considered by several people. Cramer (Ref 7) and Wald (Ref 53) showed it to be consistent under independent observations for reasonably general conditions. Maybeck (Ref 33) and Tse and Anton (Ref 50) extend this work to dependent observation sequences. The following theorem gives sufficient conditions for the maximum likelihood estimate to be consistent in probability.

Theorem 3.9

Let $\hat{\phi}(Z_k)$ denote the maximum likelihood estimate of $\phi \varepsilon \Phi$ under the observation sequence Z_n. Then the estimate

$\hat{\phi}(Z_k)$ is consistent in probability (i.e., ϕ is identifiable) if for all ϕ_1, ϕ_2 ϵ Φ, $\phi_1 \neq \phi_2$, there exists an infinite set $S \subset I^+$ (where I^+ is the set of non-negative integers) such that, for all k ϵ S, the inequality

$$p(z_k | Z_{k-1}, \phi_1) \neq p(z_k | Z_{k-1}, \phi_2)$$

is satisfied.

Theorem 3.9 is proved in Ref 50 under some rather general assumptions concerning the joint density function $p(Z_k | \phi)$.

Now return to the nonlinear model described by Eq (3.4.1) and the discrete observation process

$$z(t_i) = y(t_i) + v(t_i) \qquad i = 1, 2, \cdots k \qquad (3.5.7)$$

with $v(t_i)$ an independent Gaussian zero-mean white noise sequence with

$$E\{v(t_i)v(t_j)\} = R(t_i) \text{ for } t_i = t_j$$
$$= 0 \text{ for } t_i \neq t_j \qquad (3.5.8)$$

An observation vector Z_k is formed consisting of a sequence of observations

$$Z_k = \left[z^T(t_1) z^T(t_2) \cdots z^T(t_k) \right]^T \qquad (3.5.9)$$

with t_i ϵ $\left[t_o, t_f \right]$ for all i.

$z(t_i)$ and $v(t_i)$ are m-vectors so that Z_k has dimension km.

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The joint probability density function of the m components of $z(t_j)$ is Gaussian and is given by

$$p(z(t_j)|\phi) = \frac{1}{(2\pi)^{m/2}|R(t_j)|^{\frac{1}{2}}} \exp\{-\frac{1}{2}\left[z(t_j)-y(t_j)\right]^T$$
$$\left[R(t_j)\right]^{-1}\left[z(t_j)-y(t_j)\right]\}$$

$$(3.5.10)$$

Note that the mean of $p(z(t_j) \mid \phi)$ is given by $\dot{E}\{z(t_j)\} = E\{y(t_j) + v(t_j)\} = y(t_j)$. We should also note that the probability density of Eq (3.5.10) is conditioned on $y(t_j)$ which is dependent on the model differential equation and output model with initial conditions x_0 and input u. Also since the observation noise is independent in time

$$p(Z_k \mid \phi) = p(z(t_1), z(t_2), \cdots, z(t_k) \mid \phi)$$
$$= \prod_{j=1}^{k} p(z(t_j) \mid \phi) \qquad (3.5.11)$$

where $p(z(t_j) \mid \phi)$ is given by Eq (3.5.10).
Note that an equivalent expression of Eq (3.5.11) is

$$p(Z_k \mid \phi) = \frac{1}{(2\pi)^{km/2}|\Lambda|^{\frac{1}{2}}} \exp\{-\frac{1}{2}\left[Z_k - Y\right]^T \Lambda^{-1}\left[Z_k - Y\right]\}$$

$$(3.5.12)$$

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where
$$Z_k = \left[z^T(t_1) z^T(t_2) \cdots z^T(t_k) \right]^T$$

$$Y = \left[y^T(t_1) y^T(t_2) \cdots y^T(t_k) \right]^T$$

and

$$\Lambda = \begin{bmatrix} R(t_1) & & & 0 \\ & R(t_2) & & \\ & & \ddots & \\ 0 & & & R(t_k) \end{bmatrix} \quad mk \times mk$$

Now suppose we repeat the process on $[t_o, t_f]$ many times so that we can obtain independent repetitions of the observation process Z_k. We can form a vector Z_k^N from these independent trials of Z_k, i.e.,

$$Z_k^N = \left[Z_k^{1T} Z_k^{2T} \cdots Z_k^{NT} \right]^T \qquad (3.5.13)$$

Note that we have required that the observation noise be independent between successive trials of Z_k, so that $E\{v^k(t_j) v^k(t_j)\} = 0$ for trials i and ℓ, $i \neq \ell$. We can make a maximum likelihood estimate of ϕ based on this observation sequence as follows:

$$\hat{\phi}(N) = \max_{\phi \in \Phi} p(Z_k^1, Z_k^2, \cdots Z_k^N \mid \phi)$$

$$= \max_{\phi \in \Phi} \prod_{j=1}^{N} p(Z_k^j \mid \phi) \qquad (3.5.14)$$

where $p(Z_k^j \mid \phi)$ is given by Eq (3.5.12).

From Theorem 3.9, we know that a sufficient condition for ϕ to be identifiable is that for all $\phi_1, \phi_2 \in \Phi$ with

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$\phi_1 \neq \phi_2$, there exists an infinite set $S \subset I^+$ such that, for all $j \in S$, the inequality

$$p(z_k^j \mid z_k^{j-1}, \phi_1) \neq p(z_k^j \mid z_k^{j-1}, \phi_2) \qquad (3.5.15)$$

is satisfied. Note that since we have independent trials of the observation process Z_k, the following is equivalent to the inequality Eq (3.5.15)

$$p(z_k^j \mid \phi_1) \neq p(z_k^j \mid \phi_2) \qquad (3.5.16)$$

where $p(z_k^j \mid \phi)$ is given by Eq (3.5.12).

From the form of $p(z_k^j \mid \phi)$ in Eq (3.5.12), it is apparent that the inequality Eq (3.5.16) is equivalent to

$$\left[z_k^j - Y(\phi_1) \right]^T \Lambda^{-1} \left[z_k^j - Y(\phi_1) \right]$$
$$\neq \left[z_k^j - Y(\phi_2) \right]^T \Lambda^{-1} \left[z_k^j - Y(\phi_2) \right] \qquad (3.5.17)$$

Consequently a sufficient condition for Eq (3.5.17) to hold is that

$$Y(\phi_1) \neq Y(\phi_2)$$

for all ϕ_1, $\phi_2 \in \Phi$ with $\phi_1 \neq \phi_2$.

But this is just a statement of ϕ being identifiable on Φ under the deterministic observation process

$$Y = \left[y^T(t_1) y^T(t_2) \cdots y^T(t_k) \right]^T$$

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The foregoing discussion is proof of the following theorem:

Theorem 3.10

Given the system modeled by

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o$$

and the discrete observation process

$$z(t_i) = y(t_i) + v(t_i) \qquad i = 1, 2, \cdots k$$

where

$$y(t_i) = h(t_i, x(t_i), \phi)$$

and $v(t_i)$ is a zero mean white Gaussian sequence; then the parameter vector ϕ is identifiable on Φ (in the sense of the definition following Eq (3.5.4)) using the maximum likelihood estimate

$$\hat{\phi}(N) = \max_{\phi \varepsilon \Phi} \prod_{j=1}^{N} p(Z_k^{\ j} \mid \phi)$$

if and only if ϕ is identifiable on Φ under the deterministic observation process

$$Y = \left[y^T(t_1) y^T(t_2) \cdots y^T(t_k) \right]^T$$

Note that we have required the observation noise $\{v(t_i)\}$ to be independent in time and independent between successive trials of the observation process Z_k. That is,

for the jth trial of the observation process; i.e., $z_k^{\,j}$
we require

$$\dot{E}\{v^j(t_i)v^j(t_\ell)\} = 0 \qquad \text{for } i \neq \ell$$

and for the trials i and ℓ of the observation process, i.e.,
$z_k^{\,i}$ and $z_k^{\,\ell}$, we require

$$\dot{E}\{v^i(t_j)v^\ell(t_k)\} = 0 \qquad \text{for } i \neq \ell,\ j = k,\ \text{and } j \neq k.$$

Also, if Φ is an open neighborhood of ϕ, then Theorems 3.1, 3.2, and 3.4 can be applied to test for deterministic local identifiability. If $\Phi \subset R^s$, Theorems 3.6 and 3.7 can be applied to test for deterministic global identifiability.

The above theorem can be applied to continuous observation processes on the interval $[t_o,\ t_f]$. This can be seen by simply considering $k > \dfrac{n}{m}$ time points on this interval and then apply the above development.

It is emphasized that Theorem 3.10 can be applied only to cases of additive white noise on the observation where independent repetitions of the observation can be obtained. If there were driving process noise, the results would be substantially different. Generally, deterministic identifiability is a necessary but not sufficient condition for stochastic identifiability.

3.6 Summary

In this chapter, we have developed sufficient conditions for identifiability of nonlinear systems.

Local Identifiability. Theorems 3.1 and 3.2 state sufficient conditions for local identifiability under discrete observations. Theorem 3.4 uses a recursion relation to present alternative sufficient conditions under discrete observations. Theorems 3.3 and 3.5 state corresponding sufficient conditions for local identifiability under continuous observation processes.

Global Identifiability. Theorems 3.6, 3.7, and 3.8 extend the above theorems to state sufficient conditions for global identifiability of nonlinear systems under both discrete and continuous observation processes.

Stochastic Identifiability. In Section 3.5, for the case of additive zero mean white noise on the observation where independent repetitions of the observation can be obtained, we found that deterministic identifiability was not only necessary but also sufficient for stochastic identifiability. Theorem 3.10 summarizes this result.

Chapter IV

COMPUTATIONAL TECHNIQUES FOR PARAMETER IDENTIFICATION

In the previous chapter, we defined local and global identifiability in terms of a one-to-one correspondence between the parameter vector and the observed data. These definitions are independent of the computational technique used for identification. Typically the parameter identification problem involves estimating certain parameters based on experimental data. The general approach is to establish a criterion or cost functional which is used to choose the best values of the parameter vector elements. We will be concerned with parameter identification in which the least squares cost functional is applied. Again we consider the class of systems modeled by

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o \qquad (4.1)$$

for $t \in [t_o, t_f]$ with observations

$$y(t_i) = h(t_i, x(t_i), \phi) \qquad i = 1, 2, \cdots, k \quad (4.2)$$

For a given input function u, initial conditions x_o and observed experimental data $z(t_i)$ for all $t_i \in [t_o, t_f]$, we want to choose ϕ such that

$$J(\phi) = \sum_{i=1}^{N} \left[z(t_i) - y(t_i) \right]^T \left[z(t_i) - y(t_i) \right] \qquad (4.3)$$

is minimized.

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In this chapter we will develop some theorems con-
cerning minimization of cost functionals, relate identi-
fiability with the problem of minimizing the least squares
cost functional, and discuss two iterative computational
techniques for determining a value of ϕ which minimizes
the cost functional.

4.1 Minimization of Cost Functionals

In this section we state some definitions and theorems
which apply to the problem of finding minimum values of
cost functions. The following definitions are taken from
Ref 39:

Definition: Let $g : D \subset R^n \to R^1$. A point $x^* \in D$ is
a <u>local</u> <u>minimizer</u> of g if there is an open neighborhood S
of x^* such that for all $x \in S \cap D$, $x \neq x^*$

$$g(x) \geq g(x^*) \tag{4.1.1}$$

x^* is a <u>proper</u> <u>local</u> <u>minimizer</u> of g if strict inequality
holds in Eq (4.1.1). If Eq (4.1.1) holds for all $x \in D_0 \subset D$
and $x^* \in D_0$, then x^* is a <u>global</u> <u>minimizer</u> of g on D_0.

Definition: A point $x^* \in$ Int D (denotes interior of D)
is a <u>critical</u> <u>point</u> of $g : D \subset R^n \to R^1$ if g has a derivative
at x^* and $\partial g(x^*)/\partial x$ (denoted by $g'(x^*)$) equals zero.

Lemma 4.1

If $x^* \in$ Int D is a local minimizer of $g : D \subset R^n \to R^1$,
and $g'(x^*)$ and $\partial^2 g(x^*)/\partial x^2$ (denoted by $g''(x^*)$) exist, then

$g'(x^*) = 0$ and $g''(x^*)$ is positive semidefinite.

Lemma 4.2

Let $g : D \subset R^n \to R^1$ and suppose $g''(x)$ exists at $x^* \in \text{Int } D$. Then x^* is a proper local minimizer of g if

    (1) $g'(x^*) = 0$

and

    (2) $g''(x^*)$ is positive definite.

Lemma 4.1 and 4.2 are the well known necessary and sufficient conditions for g to have a minimum at x^*.

Definition: A functional $g : D \subset R^n \to R^1$ is convex on a convex set $D_o \subset D$ if for all $x, y, \in D_o$ and $0 < \alpha < 1$

$$g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha)g(y)$$

The functional is strictly convex on D_o if strict inequality holds whenever $x \ne y$.

Theorem 4.1

Assume that for $g : D \subset R^n \to R^1$, $g''(x)$ exist for each point of a convex set $D_o \subset D$. Then g is convex on D_o if and only if $g''(x)$ is positive semidefinite for all $x \in D_o$. Moreover, g is strictly convex on D_o if $g''(x)$ is positive definite for all $x \in D_o$.

Proof: See Ref 39:87.

From the Theorem 4.1, we see for example that the functional $g(x) = x^T A x$ is convex if and only if A is positive semi-definite and strictly convex if A is positive definite.

Theorem 4.2

Suppose that $g : D \subset R^n \to R^1$ is convex and differentiable on an open convex set $D_o \subset D$. Then $x^* \in D_o$ is a critical point of g if and only if x^* is a global minimizer on D_o. Moreover, if g is strictly convex on D_o, there is at most one critical point in D_o.

Proof: See Ref 39:101.

From Theorems 4.1 and 4.2, we can state the following corollary.

Corollary 4.1

Given $g : D \subset R^n \to R^1$. If $g''(x)$ is positive definite at each point of an open convex set $D_o \subset D$, then g has at most one local minimizer in D_o. That is, if a critical point exists in D_o, it is the only critical point in D_o and is a global minimizer.

Now consider a particular class of functionals which we will refer to as quadratic. Phillipson (42) has developed some useful fundamental results pertaining to quadratic functionals defined on a Hilbert space. For our purposes we can assume the Hilbert space to be R^n with

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inner product $(x, y) = x^T y$ and with the Euclidean norm. Let $x, y, z \in R^n$ and consider the following definitions given by Phillipson:

$\ell(x)$ is a continuous <u>linear</u> <u>functional</u> on R^n if

$$|\ell(x)| \leq k||x|| \qquad k < \infty$$

and for $\alpha, \beta \in R^1$

$$\ell(\alpha x + \beta y) = \alpha\ell(x) + \beta\ell(y)$$

$q(x, y)$ is a continuous <u>bilinear</u> <u>functional</u> on R^n if

$$|q(x, y)| \leq k||x|| \ ||y|| \qquad k < \infty$$

and for $\alpha, \beta \in R^1$

$$q(\alpha x + \beta y, z) = \alpha q(x, z) + \beta q(y, z)$$
$$q(x, \alpha y + \beta z) = \alpha q(x, y) + \beta q(x, z)$$

$h(x)$ is a <u>quadratic</u> <u>functional</u> on R^n with the property

$$h(x) = q(x, x) - 2\ell(x)$$

Lemma 4.3

$h(x)$ is a strictly convex functional on a convex set $D \subset R^n$.

Proof: From the definition of a convex functional, we need to show that

$$h(\alpha x + (1-\alpha)y) < \alpha h(x) + (1-\alpha)h(y) \qquad 0 < \alpha < 1$$

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From the definition of h(x)

$$(1-\alpha)h(y) + \alpha h(x) = (1-\alpha)q(y,y) + \alpha q(x,x)$$
$$- 2\big[(1-\alpha)\ell(y) + \alpha\ell(x)\big]$$
$$= q(y,y) + \alpha q(y, x-y) + \alpha q(x-y, y)$$
$$+ \alpha q(x-y, x-y) - 2\ell\big[(1-\alpha)y + \alpha x\big]$$

Since $\alpha q(x-y, x-y) > \alpha^2 q(x-y, x-y)$ we obtain

$$(1-\alpha)h(y) + \alpha h(x) > q(y,y) + \alpha q(y, x-y) + \alpha q(x-y, y)$$
$$+ \alpha^2 q(x-y, x-y) - 2\ell\big[(1-\alpha)y + \alpha x\big]$$
$$= q(y + \alpha(x-y), y + \alpha(x-y))$$
$$- 2\ell\big[(1-\alpha)y + \alpha x\big]$$
$$= h((1-\alpha)y + \alpha x)$$

$$\text{Q.E.D.}$$

Theorem 4.3

Let h(x) be a quadratic functional defined on a closed, convex set $D \subset R^n$. Then there exists an $x^* \in D$ which is a unique global minimizer on D. That is, there is a unique x* such that

$$h(x) \geq h(x^*) \qquad \text{for all } x \in D.$$

Proof: The existence is proved in Appendix 2.7 of Ref 42. Uniqueness follows quickly from h(x) being strictly convex. Suppose we assume there is an $x^{**} \neq x^*$ with $x^{**} \in D$ such that

$$h(x^{**}) = h(x^*) \overset{\Delta}{=} h_{min}$$

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From Lemma 4.3, we know

$$h(\alpha x^* + (1-\alpha)x^{**}) < \alpha h(x^*) + (1-\alpha)h(x^{**}) = h_{min}$$

Now let $\alpha x^* + (1-\alpha)x^{**} = y \; \varepsilon \; D$.
Then

$$h(y) < h_{min}$$

which is a contradiction.

Q.E.D.

Theorem 4.3 is a fundamental result which assures us that a unique global minimizer of $h(x)$ exists on a closed, convex set $D \subset R^n$. Two additional characteristics of the global minimizer of $h(x)$ are contained in the following theorems.

Theorem 4.4

The unique global minimizer x^* for which

$$h(x^*) \leq h(x) \qquad \text{for all } x \; \varepsilon \; D$$

is characterized by

$$q(x^*, x-x^*) \geq \ell(x-x^*)$$

for all $x \; \varepsilon \; D$.

Proof: See Ref 42:20.

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Theorem 4.5

For $D = R^n$, then the unique global minimizer, x^*, is characterized by

$$q(x^*, x) - \ell(x) = 0 \qquad \text{for all } x \in R^n.$$

Proof: See Ref 42:21.

In the next section we will apply these results to the least squares cost functional.

4.2 Identifiability and Minimization of Cost Functionals

In this section, we want to relate identifiability with the problem of minimizing the least squares cost functional given by Eq (4.3). Recall that the discrete observation process may be represented by a vector

$$Y(\phi) = \left[y^T(t_1) y^T(t_2) \cdot \cdot \cdot y^T(t_k) \right]^T \qquad (4.2.1)$$

and the corresponding experimental data by

$$Z = \left[z^T(t_1) z^T(t_2) \cdot \cdot \cdot z^T(t_k) \right]^T \qquad (4.2.2)$$

Then the least squares cost functional takes the form

$$J(\phi) = \left[Z - Y(\phi) \right]^T \left[Z - Y(\phi) \right] \qquad (4.2.3)$$

We also note that $J(\phi)$ is a quadratic functional as previously defined when viewed as a function of Y. That is

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$J : \Omega \subset R^{mk} \to R^1$ is a quadratic functional since

$$J(Y) = Y^T Y - 2\left[Z^T Y - \frac{1}{2} Z^T Z\right]$$

$$= q(Y, Y) - 2\ell(Y)$$

Theorem 4.6

Let the class of models be systems described by

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad t \in [t_o, t_f]$$

$$x(t_o) = x_o$$

and the discrete observation process be described by $Y(\phi)$ with the corresponding experimental data Z. Further let $Y(\phi) : \Phi \subset R^s \to \Omega \subset R^{mk}$ be a mapping such that Ω is a closed convex set. Then the least squares cost functional $J(\phi)$ has a unique global minimizer, $\phi^* \in \Phi$ if the system is globally identifiable on Φ.

Proof: Since $J(Y)$ is a quadratic functional, we know from Theorem 4.3 that there is a unique $Y^* \in \Omega$ which is a global minimizer of J on Ω. Also if the system is globally identifiable on Φ, then

$$Y(\phi_1) \neq Y(\phi_2)$$

for all ϕ_1, ϕ_2, $\in \Phi$ with $\phi_1 \neq \phi_2$. Therefore there is a unique ϕ^* which maps to Y^*.

Q.E.D.

Note that the requirement of a closed set Ω is needed for existence. The uniqueness follows for any convex set Ω

from the fact that J is convex on Ω. Although we proved convexity for any quadratic functional in Lemma 4.3, we could also show J(Y) is convex by observing that $d^2J(Y)/dY^2$ is positive definite on Ω and from Theorem 4.1 this implies convexity. Also, the uniqueness of a global minimizer, if it exists, on an open convex set follows immediately from Corollary 4.1.

Theorem 4.7

If ϕ is mapped <u>onto</u> R^{mk} by Y, that is $Y : \Phi \subset R^s \rightarrow R^{mk}$, then the unique global minimizer ϕ^* of J is characterized by

$$Y(\phi^*) = Z$$

Proof: From Theorem 4.5 we know that

$$Y^T(\phi^*)Y(\phi) - 2\left[Z^TY(\phi) - \frac{1}{2} Z^TZ\right] = 0$$

for all $Y \in R^{mk}$. At $Y(\phi) = Y(\phi^*)$, this implies

$$Y(\phi^*) = Z$$

Q.E.D.

The above theorem can be argued intuitively, since if one has the choice of any $Y(\phi) \in R^{mk}$, it is clear that the least squares cost functional will have its minimum value of zero at $Y(\phi^*) = Z$.

The following theorem relates local identifiability to local minimum of $J(\phi)$.

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Theorem 4.8

If $\phi^* \in \Phi$ is a critical point of the least squares cost functional $J(\phi)$ and $\partial Y(\phi^*)/\partial \phi$ (denoted by $Y'(\phi^*)$) has rank s, then ϕ^* is a unique local minimizer on an open neighborhood S of ϕ^*.

Proof: If ϕ^* is a critical point, then

$$\frac{\partial J(\phi^*)}{\partial \phi} = 0^T$$

or from Eq (4.2.3)

$$-2\left[Y'(\phi^*)\right]^T \left[Z - Y(\phi^*)\right] = 0 \qquad (4.2.4)$$

Now since $Y'(\phi^*)$ has rank s, there exists a nonsingular mk x mk matrix Q such that $Y_1'(\phi^*)$ given by

$$Y_1'(\phi^*) = Q \, Y'(\phi^*)$$

is in Hermite normal form (see Ref 37:54). The form of $Y_1'(\phi^*)$ is uniquely determined by $Y'(\phi^*)$ and will be characterized as follows:

$$Y_1'(\phi^*) = \left[\begin{array}{c} Y_2'(\phi^*) \\ \hline 0 \end{array}\right]$$

where $Y_2'(\phi^*)$ is an s x s nonsingular matrix.

Eq (4.2.4) is equivalent to the following

$$-2\left[Y'(\phi^*)\right]^T Q^T \left[Q^T\right]^{-1} \left[Z - Y(\phi^*)\right] = 0 \qquad (4.2.5)$$

where Q is the nonsingular mk x mk matrix yielding the Hermite normal form of $Y'(\phi^*)$.

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Define the following

$$\left[Y_3{}'(\phi*)\right]^T \overset{\Delta}{=} \left[Y'(\phi*)\right]^T Q^T = \left[Q \; Y'(\phi*)\right]^T$$

$$= \left[Y_1{}'(\phi*)\right]^T$$

and

$$Z_3 - Y_3(\phi*) \overset{\Delta}{=} \left[Q^T\right]^{-1}\left[Z - Y(\phi*)\right]$$

Note that

$$Y_3{}'(\phi*) = \left[\begin{array}{c} Y_4{}'(\phi*) \\ \hline 0 \end{array}\right]$$

where $Y_4{}'(\phi*)$ is $s \times s$ and nonsingular. Then Eq (4.2.5) takes the form

$$-2 \left[\begin{array}{c} Y_4{}'(\phi*) \\ \hline 0 \end{array}\right]^T \left[Z_3 - Y_3(\phi*)\right] = 0$$

or

$$-2 \left[\begin{array}{c} Y_4{}'(\phi*) \\ \hline 0 \end{array}\right]^T \left[\begin{array}{c} Z_4 - Y_4(\phi*) \\ \hline Z_5 - Y_5(\phi*) \end{array}\right] = 0$$

This yeilds

$$-2\left[Y_4{}'(\phi*)\right]^T \left[Z_4 - Y_4(\phi*)\right]$$

$$-2\left[0\right]^T \left[Z_5 - Y_5(\phi*)\right] = 0 \qquad (4.2.6)$$

The first term of Eq (4.2.6) can only have the solution

$$Z_4 = Y_4(\phi*)$$

since $Y_4{}'(\phi*)$ is nonsingular.

Recall from Theorem 3.1 that if Y has a strong derivative at $\phi*$ and $Y'(\phi*)$ has rank s, then the system is locally identifiable at $\phi*$. This means $Y(\phi)$ is one-to-one on some open neighborhood S_1 of $\phi*$ and therefore only $\phi* \in S_1$ satisfies $Y_4(\phi*) = Z_4$.

We will use the above result to show that $\partial^2 J/\partial\phi^2$ is positive definite on some neighborhood of $\phi*$. We obtain an expression for $\partial^2 J(\phi*)/\partial\phi^2$ expressed in the coordinate system resulting from the previous transformation by taking the derivative of Eq (4.2.6); i.e.,

$$\frac{\partial^2 J(\phi*)}{\partial\phi^2} = -2\{ \left[Y_4''(\phi*) \right]^T \left[Z_4 - Y_4(\phi*) \right]$$

$$- \left[Y_4'(\phi*) \right]^T \left[Y_4'(\phi*) \right] \} \qquad (4.2.7)$$

From the previous discussion we know that a neighborhood $S_2 = \{\phi \ni ||\phi - \phi*|| < \delta\}$ exists such that for $\phi \in S_2$

$$|| \left[Y_4''(\phi*) \right]^T \left[Z_4 - Y_4(\phi*) \right] || < \epsilon \quad .$$

The second term of Eq (4.2.7) can be written as

$$M_4(\phi*) \stackrel{\Delta}{=} \left[Y_4'(\phi*) \right]^T \left[Y_4'(\phi*) \right]$$

$$= \sum_{i=1}^{k} \left[\frac{\partial y_4(t_i)}{\partial \phi} \right]^T \left[\frac{\partial y_4(t_i)}{\partial \phi} \right] .$$

$M_4(\phi*)$ is a real symmetric matrix. We proved in Theorem 2.3 that $M_4(\phi*)$ is nonsingular if and only if $Y_4'(\phi*)$ has rank s. Therefore $M_4(\phi*)$ is nonsingular. Also, a real symmetric matrix of the form $M_4(\phi) = A^T A$ is positive definite if it is nonsingular (see, e.g., Ref 20:38). Therefore there is an open neighborhood S of $\phi*$ where $M_4(\phi)$ is positive definite and such that $\partial^2 J(\phi)/\partial\phi^2$ is positive definite for $\phi \in S$.

Then from Corollary 4.1 we know that $\phi*$ is a unique local minimizer of J on S.

Q.E.D.

Theorem 4.9

If $\phi*$ is a critical point of the least squares cost functional $J(\phi)$ and $Y'(\phi*)$ has rank s for a system which is globally identifiable on an open set $\Phi \subset D$, then $\phi*$ is a unique global minimizer of J on Φ.

Proof: Following the same procedure as the proof for Theorem 4.8, we can show that $\phi*$ is the only critical point of J on Φ. That is, the necessary condition for a critical

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point, $\partial J(\phi^*)/\partial\phi = 0$, has a unique solution $Y(\phi^*)$. Since the system is globally identifiable on Φ, then there is only one ϕ^* which can be mapped to $Y(\phi^*)$. To show that ϕ^* is a minimizer of J, we can use exactly the same arguments as in the previous theorem to show that ϕ^* is a local minimizer on some open neighborhood of ϕ^*. Then since ϕ^* is the only critical point on Φ, it necessarily must be a global minimizer on Φ.

<div align="right">Q.E.D.</div>

The above discussion illustrates that identifiability and minimization of the least squares cost functional are intimately related. Theorem 4.8 has given us a very useful result. If through some manner we can find a critical point ϕ^* of the least squares cost functional J, then if $Y'(\phi^*)$ has rank s, we not only assure local identifiability but also assure that ϕ^* is a local minimizer of J. We also know that $Y'(\phi)$ has rank s if and only if

$$M(\phi) = \left[Y'(\phi^*)\right]^T \left[Y'(\phi^*)\right]$$

is nonsingular.

In the next section we will discuss computation techniques for solving the equation $\partial J/\partial\phi = 0^T$, the solution of which, of course, defines a critical point.

4.3 Computational Techniques

Two computational techniques will be briefly presented for minimizing the least squares cost functional. A basic

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gradient method will be developed, followed by a quasi-linearization technique.

4.3.1 Gradient Method

The gradient method is an iterative technique in which each step is in the direction of steepest decent toward the minimum of the cost functional. We know from Lemma 4.1 that a necessary condition for a minimum is

$$\frac{\partial J}{\partial \phi} = 0^T \tag{4.3.1}$$

The vector $\left[\frac{\partial J}{\partial \phi}\right]^T$ is the gradient vector and Eq (4.3.1) is a set of s equations which, in general, may be very difficult to solve. This motivates the development of an iterative method. The gradient technique is an iterative method based on the Taylor expansion

$$J(\phi) = J(\phi^k) + \frac{dJ(\phi^k)}{d\phi^k} (\phi - \phi^k) + \cdots \tag{4.3.2}$$

where ϕ^k is an assumed value of ϕ. Define ΔJ^k by

$$\Delta J^k = J(\phi) - J(\phi^k) \tag{4.3.3}$$

so that to first order

$$\Delta J^k \cong \frac{dJ(\phi^k)}{d\phi^k} \Delta \phi^k \tag{4.3.4}$$

where $\Delta \phi^k = \phi - \phi^k$.

In the gradient technique, the objective is to pick $\Delta \phi^k$ such that ΔJ^k is as negative as possible and thereby

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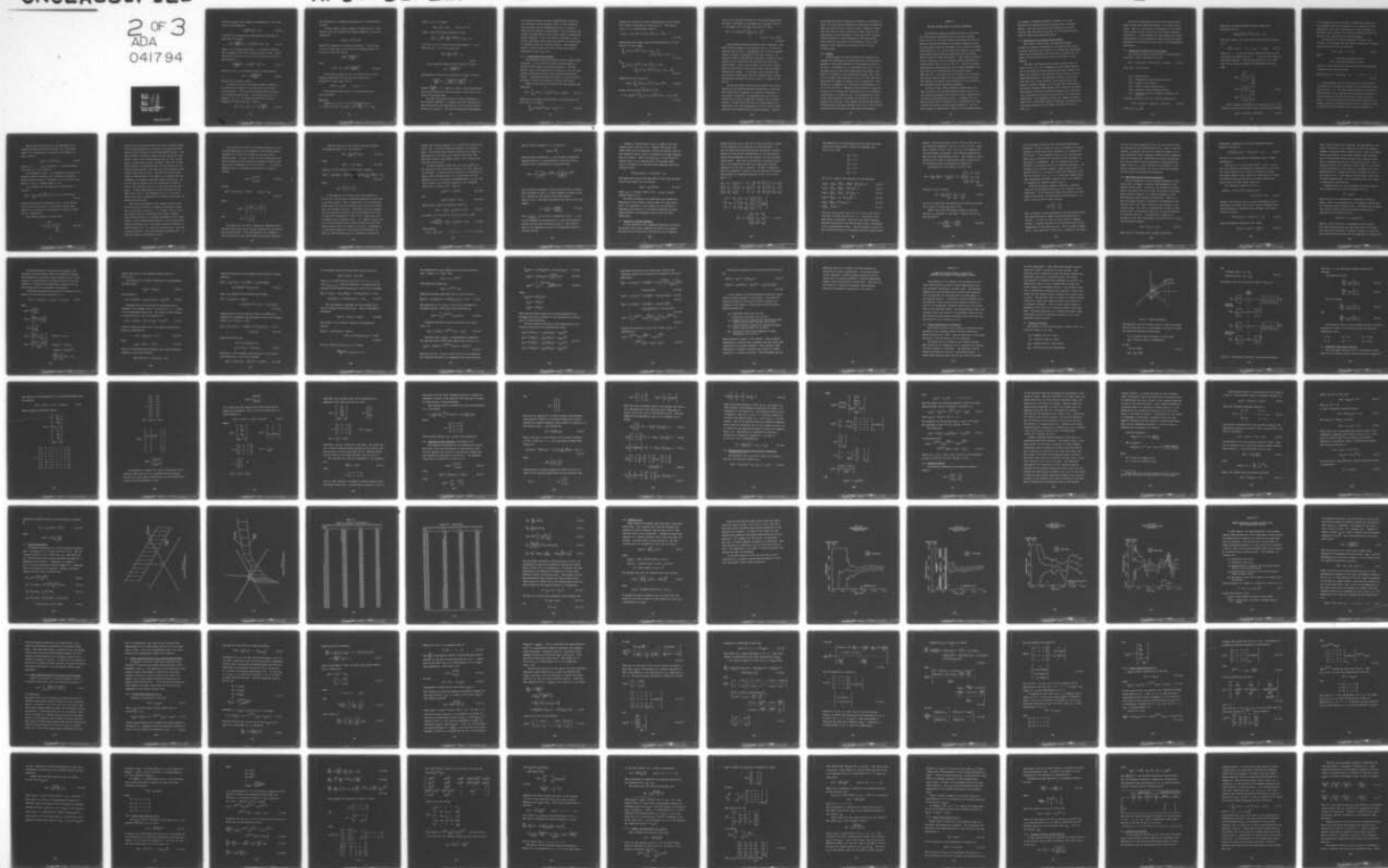
make the largest step towards the minimum of J. The step
size is constrained by

$$\left[\Delta\phi^k\right]^T \left[\Delta\phi^k\right] = q^2 \qquad (4.3.5)$$

To exploit the unconstrained optimization technique, we
will want to minimize

$$\Delta J^k = \frac{dJ(\phi^k)}{d\phi^k} \Delta\phi^k + \nu\{\left[\Delta\phi^k\right]^T \left[\Delta\phi^k\right] - q^2\} \qquad (4.3.6)$$

where ν is a Lagrange multiplier. A necessary condition
for a minimum is the first variation must be zero. Taking
the first variation of Eq (4.3.6) and setting it equal to
zero results in

$$\{\left[\frac{dJ(\phi^k)}{d\phi^k}\right] + 2 \nu\left[\Delta\phi^k\right]^T\} \, \delta\Delta\phi^k = 0 \qquad (4.3.7)$$

From Eq (4.3.7), it is apparent that we should choose

$$\Delta\phi^k = - K^k\left[\frac{dJ(\phi^k)}{d\phi^k}\right]^T \qquad (4.3.8)$$

where K^k is a positive scalar.

Eq (4.3.8) tells us the largest possible change in J
is obtained by stepping in the direction of the local
gradient vector. Thus the iterative calculation proceeds
in a straightforward manner. An initial value of ϕ is
assumed; i.e., ϕ^1. We calculate $dJ(\phi^1)/d\phi^1$ and then
determine ϕ^2 from Eq (4.3.8). That is

$$\phi^{k+1} = \phi^k + \Delta\phi^k = \phi^k - K^k \frac{dJ(\phi^k)}{d\phi^k} \qquad (4.3.9)$$

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The calculation is repeated to converge on a critical point of $J(\phi)$.

In some cases it may be useful to choose the $(k + 1)$-st iterate along the direction of steepest descent in the norm defined by

$$||x||_k = (x^T N^k x)^{\frac{1}{2}}$$

where N^k is symmetric and positive definite. In this case it can be shown that the direction of steepest descent of J is given by (see Ref 20:245)

$$- \left[N^k\right]^{-1} \left[\frac{dJ(\phi^k)}{d\phi^k}\right]^T$$

Then

$$\phi^{k+1} = \phi^k - \left[N^k\right]^{-1}\left[\frac{dJ(\phi^k)}{d\phi^k}\right]^T \qquad (4.3.10)$$

The gradient method is part of a broad class of minimization algorithms in which the cost functional is decreased at each stage; that is

$$J(\phi^{k+1}) \leq J(\phi^k) \qquad k = 0, 1, \cdots$$

The following lemma relates to the convergence of such techniques.

Lemma 4.4

Suppose $J : D \subset R^s \rightarrow R^1$ is differentiable at $x \varepsilon \text{ Int}(D)$ and that, for some $p \varepsilon R^s$, $\left[\frac{\partial J}{\partial x}\right] p > 0$. Then

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there is a $\delta > 0$ so that

$$J(\phi - \alpha p) < J(\phi) \qquad \text{for } \alpha \; \epsilon \; (0, \; \delta).$$

Proof: From the differentiability we know

$$\lim_{\alpha \to 0} \left[\frac{J(\phi - \alpha p) - J(\phi)}{\alpha} \right] + J'(\phi) \; p = 0$$

If $J'(\phi) \; p > 0$, it follows that we may choose a $\delta > 0$ so that for all $\alpha \; \epsilon \; (0, \; \delta)$ we will have

$$\frac{J(\phi - \alpha p) - J(\phi)}{\alpha} \; < \; 0$$

<div align="right">Q.E.D.</div>

In the gradient method we have chosen p^k so that

$$p^k = \left[\frac{dJ(\phi^k)}{\phi^k} \right]^T$$

and thereby we satisfy the conditions of Lemma 4.4 since

$$\frac{dJ(\phi^k)}{d\phi^k} \; p^k = \left[\frac{dJ(\phi^k)}{d\phi^k} \right] \left[\frac{dJ(\phi^k)}{d\phi^k} \right]^T > 0$$

whenever $\frac{dJ(\phi^k)}{d\phi^k} \neq 0$. Lemma 4.3 tells us that the gradient method will reduce the cost functional at each step if we choose K^k suitably small.

The above discussion reveals a primary advantage of the gradient technique. It reduces the cost functional at each step without any requirement for the initial guess to be near the solution. Also the computational requirements

are relatively small; the main computational problem is the evaluation of the gradient vector at each iteration. This technique has the disadvantage that it converges slowly as the solution is approached. This is because the gradient becomes small as the critical point is approached. Another drawback is that the gradient technique is relatively inefficient when ridges and ravines are encountered in stepping toward the optimal solution. The technique tends to "walk" between the sides of the ravine while progressing slowly out of it.

4.3.2 Quasilinearization Method

The preceding section developed a direct computational technique. In this section we will briefly examine an indirect method known as quasilinearization. Indirect methods result from attacking the solution to the two point boundary problem which arise from applying optimization theory. The quasilinearization method is also an iterative procedure, initially developed by Bellman and Kalaba.

Again we are trying to minimize the least squares cost functional

$$J(\phi) = \sum_{i=1}^{N} \left[z(t_i) - y(t_i) \right]^T \left[z(t_i) - y(t_i) \right] \qquad (4.3.11)$$

Applying the necessary condition for a critical point, we set $\partial J / \partial \phi = 0$ to obtain

$$\sum_{i=1}^{N} \left[y'(t_i) \right]^T \left[z(t_i) - y(t_i) \right] = 0 \qquad (4.3.12)$$

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Assume that we have an initial approximation for the parameter vector, which again is denoted by ϕ^k. Then expand $y(t_i, \phi)$ in a Taylor series about ϕ^k.

$$y(t_i) = y(t_i, \phi^k) + [y'(t_i, \phi^k)] [\phi - \phi^k] + \cdots$$

$$(4.3.13)$$

Neglecting higher order terms and substituting Eq (4.3.13) into Eq (4.3.12) yields

$$\sum_{i=1}^{N} [y'(t_i, \phi^k)]^T \{z(t_i) - y(t_i, \phi^k) - y'(t_i, \phi^k)[\phi - \phi^k]\} = 0$$

$$(4.3.14)$$

or

$$\sum_{i=1}^{N} [y'(t_i, \phi^k)]^T [y'(t_i, \phi^k)] [\phi - \phi^k]$$

$$= \sum_{i=1}^{N} [y'(t_i, \phi^k)]^T [z(t_i) - y(t_i, \phi^k)]$$

$$(4.3.15)$$

Assuming that the inverse of

$$M(\phi^k) = \sum_{i=1}^{N} [y'(t_i, \phi^k)]^T [y'(t_i, \phi^k)]$$

$$(4.3.16)$$

exists, we can obtain from Eq (4.3.15)

$$\phi - \phi^k = [M(\phi^k)]^{-1} \sum_{i=1}^{N} [y'(t_i, \phi^k)]^T [z(t_i) - y(t_i, \phi^k)]$$

$$(4.3.17)$$

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Eq (4.3.17) provides the basis for the quasilinearization iterative technique for converging to a critical point of J. We assume a ϕ^k, and then calculate ϕ^{k+1} from

$$\phi^{k+1} = \phi^k + \left[M(\phi^k) \right]^{-1} \sum_{i=1}^{N} \left[y'(t_i, \phi^k) \right]^T$$

$$\left[z(t_i) - y(t_i, \phi^k) \right]$$

(4.3.18)

The quasilinearization algorithm often requires a very good initial estimate of the unknown vector ϕ in order to converge (Ref 44:152). One approach is to use the gradient method to obtain an estimate ϕ^k near the solution. As discussed before, the gradient technique is capable of converging to the vicinity of the solution starting from a poor estimate. Once a good estimate is obtained, convergence to the solution can be quite rapid using the quasilinearization method. On the other hand, examples have been given to demonstrate that the algorithm may, however, diverge when started arbitrarily close to a solution (see Ref 4).

It was noted above that the matrix $M(\phi^k)$ must have an inverse for the quasilinearization algorithm to function. This matrix may be singular, or nearly so, especially in the initial iterations. As discussed earlier, the non-singularity of $M(\phi)$ is required to assure local identifiability. We also found that for the least squares cost functional, that the nonsingularity of $M(\phi)$ will be sufficient for the solution to be a local minimizer. Thus if

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the quasilinearization technique converges to a solution of $\partial J/\partial \phi = 0$, then both the necessary and sufficient conditions have been satisfied for a local minimum. Banks and Groome (Ref 43) arrive at this conclusion in their study of the convergence of this algorithm. We can make this statement based on Theorem 4.8 which depends on the least squares cost functional and system identifiability, but is independent of the computational technique used to find a critical point.

4.4 Summary

In this chapter we have developed some theorems concerning the minimization of cost functions, with particular emphasis on the least squared cost functional as given in Eq (4.3). In Theorem 4.8, it was shown that if $\phi^* \in \Phi$ is a critical point of the least squares cost functional, $J(\phi)$, which is a function of the observation process $Y(\phi)$, then ϕ^* is a unique local minimizer of $J(\phi)$ if $Y'(\phi^*)$ has rank s. Recalling the results of Chapter III, we know that ϕ, the vector of s parameters, is also locally identifiable at ϕ^* if $Y'(\phi^*)$ has rank s. Theorem 4.8 was extended to apply to an open set Φ by adding the requirement that ϕ be globally identifiable on Φ (Theorem 4.9). Two computational techniques were presented as methods for minimizing the least squares cost function. The gradient and quasilinearization techniques were developed together with some of the features which characterize these two computational procedures.

Chapter V

OPTIMAL CONTROL MODEL FOR HUMAN PERFORMANCE

The previous chapters developed sufficient conditions for establishing identifiability of nonlinear systems. Also, the use of the least squares cost functional and the maximum likelihood technique to estimate parameters based on experimental data was discussed in the context of system identifiability. We want to apply this theory to the parameter identification problem associated with the optimal control model for human performance. This model has proven to be an effective tool for modeling human performance in various tasks (Refs 5, 20, 21, 22, 23). Previous attempts to examine the identifiability of the model parameters have been limited to investigations using linear theory (Refs 40, 41). The purpose of this chapter is to describe the optimal control model for human performance and place the problem of parameter identifiability in the context of the nonlinear theory previously presented.

As will be seen in the subsequent development, the basic structure of the optimal control model results in a set of model parameters, the values of which require identification using experimental data. In Chapter VII, we will use the theory of Chapter III to investigate the identifiability of the uncertain model parameters. We will also attempt to identify the values of the parameters using

the gradient technique discussed in Chapter IV in conjunction with experimental data obtained from a simulator system. The next two chapters will develop the optimal control model and relate the model to the simulator system used for obtaining experimental data.

5.1 Structure of the Optimal Control Model

The optimal control model for human performance assumes the operator performance is dictated by the desire to behave optimally with respect to a chosen cost functional. Factors are incorporated into the model to account for the human's inherent limitations such as time delay and randomness.

The basic structure of the optimal control model is shown in Fig. 1.1. It is assumed that the system dynamics can be described by linear equations and these dynamics are under the influence of an operator control input as well as random disturbances. The operator observes a set of outputs that are a linear function of the system state and operator control; however, it is assumed that the operator perceives a delayed noisy version of the observed variables. The operator's control is derived assuming that a quadratic cost functional is minimized on the basis of the perceived information. The foregoing scenario for human performance gives rise to the Kalman filter, predictor, and optimal control gains appearing in the model structure as shown in Fig. 1.1.

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We will be concerned with using the optimal control model to represent human performance in a tracking situation. That is, the operator is attempting to accurately track a moving target with the aid of certain displayed information. In the remainder of this chapter, the optimal control model will be described in some detail, with model parameters being defined as they arise in the mathematical development.

5.2 Mathematical Description of the Model

We assume the system dynamics can be represented by a stochastic linear differential equation

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \Gamma(t)d\beta(t) \qquad (5.2.1)$$

with $x(t_o) = x_o$ and where

$x(t)$ = system state

$A(t)$ = homogeneous system dynamics matrix

$B(t)$ = control input matrix

$u(t)$ = human's scalar control input

$\Gamma(t)$ = process noise distribution matrix

$\beta(t)$ = Brownian motion of diffusion strength $W(t)$ for all t

Proceeding formally, Eq (5.2.1) can be expressed as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Gamma(t)w(t) \qquad (5.2.2)$$

with $x(t_o) = x_o$ and

where $w(t)$ is zero-mean white Gaussian noise with covariance kernel

$$\dot{E}\{w(t_1)w^T(t_2)\} \overset{\Delta}{=} W(t_1)\delta(t_1 - t_2)$$

The state vector, $x(t)$, can be partitioned into two parts; i.e.,

$$x^T(t) = \begin{bmatrix} x_1 \cdot \cdot \cdot x_\ell & \vdots & x_{\ell+1} \cdot \cdot \cdot x_{\ell+n} \end{bmatrix} \qquad (5.2.3)$$

where $x_1, \cdot \cdot \cdot x_\ell$ are the states associated with the disturbance noise model and $x_{\ell+1}, \cdot \cdot \cdot x_{\ell+n}$ are the states of the system dynamics. Then the matrices $A(t)$, $\Gamma(t)$, and $B(t)$ can be viewed as having the following partitioned form

$$A(t) = \begin{bmatrix} A_n & \vdots & 0 \\ \text{---} & \vdots & \text{---} \\ A_B & \vdots & A_d \end{bmatrix} \begin{matrix} \ell \times \ell \\ \\ \\ \\ n \times n \end{matrix}$$

$$\Gamma(t) = \begin{bmatrix} \Gamma_n(t) \\ \text{---} \\ 0 \end{bmatrix} \begin{matrix} \ell \times 1 \\ \\ n \times 1 \end{matrix} \qquad (5.2.4)$$

$$B(t) = \begin{bmatrix} 0 \\ \text{---} \\ B_c(t) \end{bmatrix} \begin{matrix} \ell \times 1 \\ \\ n \times 1 \end{matrix}$$

The m variables displayed to the operator, $y_d(t)$, are linearly related to the system state and control as follows

$$y_d(t) = C(t)x(t) + D(t)u(t) \qquad (5.2.5)$$

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It is assumed that if a variable is displayed visually to the operator, then rate of change of that quantity is also perceived by him (32). Therefore if a visual display is utilized, $y_d(t)$ includes all variables displayed explicitly plus the first derivatives of those variables.

To account for the human operator's inherent limitations, it is assumed that the operator perceives a delayed and noisy version of $y_d(t)$ given by

$$y(t) = y_d(t - \tau) + v(t) \qquad (5.2.6)$$

where

$$\tau = \text{equivalent perceptual delay}$$
$$v(t) = \text{equivalent observation noise}$$

with $v(t)$ a zero-mean white Gaussian noise with independent components and variance kernels

$$E\{v_i(t_1)v_i(t_2)\} = V_i(t_1)\delta(t_1 - t_2) \qquad i = 1, 2, \cdots, m$$
$$(5.2.7)$$

The randomness that the human injects into the system (referred to as remnant) is included in the equivalent observation noise, $v(t)$. In other words, sources of remnant are referred back to the input of the human model and appear as noise associated with the observation of displayed variables. It is very difficult to differentiate among various remnant sources, and combining them into an equivalent observation noise has been found to be a valid procedure (5).

When directly viewing $y_i(t)$, the magnitude of the associated observation noise variance has been found (Refs 5, 23) empirically to scale with the variance of $y_i(t)$; that is

$$V_i(t) = \rho_i \; \dot{E}\{y_i^2(t)\} \qquad (5.2.8)$$

where ρ_1, ρ_2, $\cdot$ $\cdot$ $\cdot$ ρ_m are constants. A typical nominal value for ρ_i is .01 π (Ref 23).

The perceptual delay, τ, is intended to represent the combined effects of sensory and information processing delays within the human. Typically, τ has a nominal value between .15 and .3 seconds (Refs 5, 23).

It is assumed that the operator is attempting to minimize

$$J_m(u) = \lim_{t_f \to \infty} \dot{E}\{ \frac{1}{t_f} \int_o^{t_f} [x^T(t) \; Q \; x(t) + g(t)\dot{u}^2(t)] dt\}$$

$$(5.2.9)$$

based on the perceived information $y(t)$. This operator cost function was chosen for its physical appeal when applied to a tracking task and because it leads to mathematical tractability.

The weighting matrix Q has the form

$$Q = \begin{bmatrix} \overset{\ell x \ell}{0} & \vdots & 0 \\ \text{----} & \vdots & \text{----} \\ 0 & \vdots & Q_d \end{bmatrix} \qquad (5.2.10)$$

$$nxn$$

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With this form the first term of the cost functional represents a mean squared error criterion with Q_d specifying a constant cost weighting of the system dynamic states. The term $g(t)\dot{u}^2(t)$ is used to account for the operator's limitation on the rate of control motion and, as such, introduces a "neuro-motor" lag in the model. This will be discussed further in association with the calculation of the optimal control gains. Note that the cost functional could include terms such as $u^2(t)$. The form of the cost functional is task-dependent and needs to be analyzed from the standpoint of operator intent and system limitations. As will be seen in Chapter VII, the experimental simulator used in this research, as well as the intent of the operators, is best represented by a cost functional of the form in Eq (5.2.9). The assumption of $t_f \rightarrow \infty$ is valid so long as the total tracking time t_f is much greater than the system time constants.

The remaining elements of the optimal control model are the result of solving a stochastic regulator problem with a time delay constraint. It is shown in Ref 26 that the optimal control is generated by a linear feedback system which consists of a cascade combination of a Kalman filter, a least mean-squared predictor, and a deterministic optimal controller gain. The Kalman estimator generates a best estimate, $\hat{x}(t - \tau)$, from the perceived data. Then the predictor obtains a least mean-squared prediction $\hat{x}(t \mid \sigma)$ from the output of the Kalman filter.

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The controller portion of the model consists of a set
of gains operating on the estimates of the system states,
followed by a simple first-order lag to model neuro-
muscular delay. As shown in Ref 26, the optimal gains are
those obtained from the noise-free optimal regulator prob-
lem and are independent of the time delay, τ. To find the
feedback gains, an augmented state vector is formed as
follows

$$x_R(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad (5.2.11)$$

so that

$$\dot{x}_R(t) = A_R(t)x_R(t) + B_R\dot{u}(t) \qquad x_R(t_o) = x_{Ro}$$

$$(5.2.12)$$

where

$$A_R(t) = \begin{bmatrix} A(t) & \vdots & B(t) \\ \text{------------} & \text{------------} \\ 0 & \vdots & 0 \end{bmatrix}$$

and

$$B_R = \begin{bmatrix} 0 \\ \text{-----} \\ 1 \end{bmatrix}$$

The subscript R is used to denote the augmented state
equation which leads to the Riccati equation for determining
the optimal control. This distinguishes this augmented
state equation from one which appears below with subscript a.

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From the solution to the linear regulator problem, the feedback gains, $\lambda(t)$, are given by

$$\lambda(t) = \frac{1}{g(t)} B_R^T P_R(t) \qquad (5.2.13)$$

where

$$\dot{u}(t) = -\lambda(t)\,\hat{x}_R(t) \qquad (5.2.14)$$

and $P_R(t)$ is the solution to the Riccati equation

$$-\dot{P}_R(t) = P_R(t)A_R(t) + A_R^T(t)P_R \cdots - P_R(t)B_R \frac{1}{g(t)} B_R^T P_C(t) + Q_R$$

$$(5.2.15)$$

where

$$Q_R = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array} \right]$$

In the general time varying case, the solution to Eq (5.2.15) is obtained by computing backwards in time from t_f to obtain $P_R(t)$. This is unrealistic in the human modeling context in which the operator does not know a priori the time variation of $A_R(t)$. Therefore it will be assumed that the operator determines the gains adaptively on line by using his information at time t (i.e., $A_R(t)$ and $g(t)$) to compute $P_R(t)$ backwards from $t = t_f$. That is, $A_R(t)$ and $g(t)$ will be assigned their current values and $P_R(t)$ computed assuming they are constant on $[t, t_f]$. We assume t_f is very large compared to the system time constants, so that $t_f \to \infty$ and therefore $P_R(t)$ is obtained by solving the

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steady state Riccati equation (i.e., solve Eq (5.2.15) with $\dot{P}_R(t) = 0$). With this adaptive procedure, the solution $P_R(t)$ of the steady state Riccati equation will change at each point in time since $A_R(t)$ and $g(t)$ vary with time. This will result in the control gains, $\lambda(t)$, varying with time (see Eq (5.2.13)).

It was previously stated that the term $g(t)\dot{u}^2(t)$ in the operator's cost functional results in a first order lag in the model. To show how this evolves, it is necessary to define three new quantities; (1) a first-order lag time constant, τ_n; (2) an operator commanded control, $u_c(t)$; and (3) the gains $\lambda_c(t)$ which are applied to the estimated states $\hat{x}(t)$ to obtain $u_c(t)$. That is

$$u_c(t) \triangleq - \lambda_c(t)\hat{x}(t) \qquad (5.2.16a)$$

and

$$u_c(t) \triangleq \tau_n\dot{u}(t) + u(t) \qquad (5.2.16b)$$

Combining the above two equations results in

$$\dot{u}(t) = - \frac{1}{\tau_n}\left[\lambda_c(t)\hat{x}(t) + u(t)\right] \qquad (5.2.17)$$

To relate $\lambda_c(t)$ to $\lambda(t)$, we use Eq (5.2.14) to show

$$- \lambda(t)\begin{bmatrix}\hat{x}(t)\\u(t)\end{bmatrix} = - \frac{1}{\tau_n}\left[\lambda_c(t)\hat{x}(t) - u(t)\right] \qquad (5.2.18)$$

which implies

$$\lambda_i(t) = \frac{1}{\tau_n} \lambda_{ci}(t) \qquad i = 1, 2, \cdots, \ell + n \quad (5.2.19)$$

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and the $(\ell+n+1)$ component of λ is given by

$$\lambda_{\ell+n+1} = \frac{1}{\tau_n} \qquad (5.2.20)$$

With the above definitions, τ_n may be given the physical interpretation of a "neuro-muscular" delay since these results yield a configuration as depicted below:
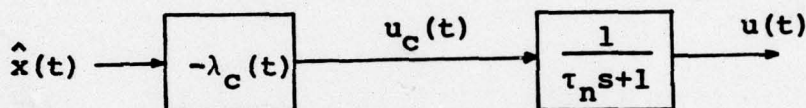
From a physical standpoint, this gives motivation to structure the model so that τ_n is held constant (a nominal value of .1 second has been found to give good results (Refs 5, 23)). Note that from Eqs (5.2.13) and (5.2.20) one obtains

$$\tau_n = \frac{1}{\lambda_{\ell+n+1}} = \frac{g(t)}{P_{R(\ell+n+1)}(t)} \qquad (5.2.21)$$

where $P_{R(\ell+n+1)}(t)$ is the $(\ell+n+1)$ component of $P_R(t)$. τ_n can be held constant by adjusting $g(t)$ at each time instant to account for the time variations of $P_R(t)$. This will be done in our computations and will be discussed further in the next section.

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Finally a "motor-noise," $v_m(t)$ is added to the commanded control (see Fig. 1.1). Without this noise, the model would allow the operator to know the precise value of the commanded control $u_c(t)$; a capability the human operator does not possess. Thus, the noise $v_m(t)$ is one source of remnant which is not referred back to the model input. $v_m(t)$ is assumed to be zero-mean white Gaussian noise with variance kernal

$$E\{v_m(t_1)v_m(t_2)\} = V_m(t_1)\delta(t_1 - t_2)$$

The magnitude of $V_m(t)$ has been found to vary with the variance of the control $u(t)$ as follows

$$V_m(t) = \rho_m\, E\{u^2(t)\} \qquad (5.2.22)$$

where ρ_m is a constant (Refs 5, 23). ρ_m has a typical nominal value of .01.

The above discussion has summarized the mathematical description of the optimal control model for human operators. The remainder of this chapter will be devoted to discussing the solution to the Riccati equation and developing equations for propagating mean states and state covariances.

5.3 Solution to Riccati Equation

In this section we will discuss an approach to solving the steady state Riccati equation and some of the characteristics of this equation as it pertains to our problem.

Recall from Eqs (5.2.3) and (5.2.4) that the first ℓ states of the augmented state equation are uncontrollable. It will be shown below that partitioning the Riccati equation results in the solution corresponding to the controllable states being decoupled from the solution for the uncontrollable states. Note that it is possible to formulate $A_R(t)$ and a cost functional so that the steady state Riccati equation will not have a solution. We show in Appendix C that our formulation results in the existence of a solution. Now consider the following partition of Eq (5.2.15) with $\dot{P}_R(t) = 0$:

$$
\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \begin{bmatrix} A_n & 0 & 0 \\ A_b & A_d & B_c \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_n^T & A_b^T & 0 \\ 0 & A_d^T & 0 \\ 0 & B_c^T & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_d & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \begin{bmatrix} 0 \\ \text{--} \\ 0 \\ \text{--} \\ 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{g(t)} \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & | & 0 & | & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} = 0 \qquad (5.3.1)
$$

The dimensions of the partitions are such that the noise
states and system dynamic states are separated (see
Eq (5.2.4)); thus

$$A_n : \ell \times \ell$$

$$A_b : n \times \ell$$

$$A_d : n \times n$$

$$B_c : n \times 1$$

$$Q_d : n \times n$$

Eq (5.3.1) leads to the following set of equations:

$$P_{11}A_n + P_{12}A_b + A_n^T P_{11} + A_b^T P_{12}^T - \frac{1}{g} P_{13}P_{13}^T = 0 \tag{5.3.2}$$

$$P_{12}A_d + A_n^T P_{12} + A_b^T P_{22} - \frac{1}{g} P_{13}P_{23}^T = 0 \tag{5.3.3}$$

$$P_{12}B_c + A_n^T P_{13} + A_b^T P_{23} - \frac{1}{g} P_{13}P_{33} = 0 \tag{5.3.4}$$

$$P_{22}A_d + A_d^T P_{22} + Q_d - \frac{1}{g} P_{23}P_{23}^T = 0 \tag{5.3.5}$$

$$P_{22}B_c + A_d^T P_{23} - \frac{1}{g} P_{23}P_{33} = 0 \tag{5.3.6}$$

$$2P_{23}^T B_c - \frac{1}{g} P_{33}^2 = 0 \tag{5.3.7}$$

With $g(t)$ and Q_d known, Eqs (5.3.5), (5.3.6), and (5.3.7)
represent a system of $[n(n+1)/2 + n+1]$ equations in the
same number of unknowns. Note also that these equations
generate the gains for the controllable states, independent
of the uncontrollable states. Thus the optimal control for
the controllable states is decoupled from the uncontrollable

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states. After solving Eqs (5.3.5), (5.3.6), and (5.3.7), the remaining equations, (5.3.2), (5.3.3), (5.3.4), represent a system of $[\ell(\ell + 1)/2 + (\ell \times n) + \ell]$ equations in the same number of unknowns. This solution produces the gains for the uncontrollable states. This method of partitioning will be used in solving the Riccati equation for the simulator system as discussed in Chapter VI.

One should also note from Eq (5.2.13) that

$$\lambda(t) = \frac{1}{g(t)} B_R^T P_R(t) = \frac{1}{g(t)} \begin{bmatrix} 0 & \vdots & 0 & \vdots & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix}$$

(5.3.8)

From Eq (5.3.8) we obtain

$$\lambda(t) = \frac{1}{g(t)} \begin{bmatrix} P_{13}^T & \vdots & P_{23}^T & \vdots & P_{33} \end{bmatrix}$$

(5.3.9)

$$\quad\quad\quad\quad 1x\ell \quad\quad 1xn \quad\quad 1x1$$

Eq (5.3.9) shows that one needs only to solve for the last row of $P_R(t)$ to obtain the gains $\lambda(t)$.

Recall that there is physical analog to motivate holding τ_n a constant in our model. From Eq (5.2.21) τ_n was obtained by

$$\tau_n = \frac{g(t)}{P_{R(\ell+n+1)}(t)} = \frac{g(t)}{P_{33}(t)}$$

(5.3.10)

τ_n can be held constant by the following computational procedure. At t_o, a value of $g(t_o)$ is assumed, the Riccati equation is solved for $P_R(t_o)$ and then τ_n is computed by Eq (5.3.10). If the desired value of τ_n is not obtained, then $g(t_o)$ is adjusted (an increase in $g(t)$ increases τ_n) and $P_R(t_o)$ is recomputed. This is repeated until the desired τ_n is obtained within the specified tolerance. The solution of the Riccati equation, $P_R(t_o)$ which gives the desired value of τ_n is used to obtain the controller gains for the interval t_o to $t_o + \Delta t$. At the next computational time (integration period Δt seconds later), the procedure is repeated with updated values of $A_R(t)$.

Solving the Riccati equation requires a specified value for the weighting matrix Q_d in the operator's cost functional. In the tracking problem the operator is interested in minimizing the mean-squared tracking error. In such a case, Q_d can be given the form

$$Q_d = \begin{bmatrix} 0 & | & 0 \\ -- & | & -- \\ 0 & | & q \end{bmatrix}$$

where q weights a state representing the difference between the desired position (target) and the actual position (sight). It turns out that for this class of problems, holding τ_n constant results in the controller gains being independent of the magnitude of q. This is shown in Appendix D. Thus specifying a value for τ_n results in the ratio

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of g(t) to q being adjusted so that the same control gains
are obtained for any positive value of q. Thus we see that
holding τ_n constant not only has a physical appeal, but is
also significant from a parameter identification standpoint.
With this formulation, identification problems associated
with the controller gains are reduced to identifying the
proper value of τ_n. This completes the discussion of the
Riccati equation solution; the next section will consider
the mean state and covariance propagation.

5.4 Mean State and Covariance Propagation

We are interested in using ensemble averages of simu-
lator data to attempt to identify the parameters of the
optimal control model. Therefore, the mathematical formu-
lation must permit use of the model to predict the average
human operator response in a tracking problem in which
deterministic target motions are specified. That is to
say, the target motion is dictated by a deterministic tra-
jectory that is not known a priori by the operator in the
tracking loop. This will be accomplished by using a tech-
nique similar to Kleinman's (Refs 20, 23) and as discussed
by Jazwinski (Ref 16). A determinsitic component is
included in the input disturbance w(t) of Eq (5.2.1) so that
the disturbance becomes

$$w(t) = w_1(t) + w_2(t) \qquad (5.4.1)$$

where $w_1(t)$ is zero-mean white Gaussian noise with

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independent components so that the covariance kernel is diagonal and given by

$$E\{w_1^T(t_1)w_1(t_2)\} = W_1(t_1)\delta(t_2 - t_1) \qquad (5.4.2)$$

and $w_2(t)$ is a deterministic disturbance such as target acceleration.

From Eg (5.4.1) we see that $E\{w(t)\} = w_2(t)$. As we will see below, this deterministic component will give rise to a mean tracking error, which is a function of time. This mean response is the result one would expect to obtain by ensemble averaging the results of many experimental trials using the same target trajectory.

The covariance kernel for $w(t)$ is

$$E\{[w(t_1) - E\{w(t_1)\}]^T [w(t_2) - E\{w(t_2)\}]\}$$
$$= W_1(t_1)\delta(t_1 - t_2) \qquad (5.4.3)$$

However, the operator has no a priori knowledge of $w_2(t)$. Therefore, he assumes (for lack of better knowledge) that the disturbance is zero-mean white Gaussian noise with a covariance kernel

$$E\{w^T(t_1)w(t_2)\} = W(t_1)\delta(t_1 - t_2) \qquad (5.4.4)$$

where the diagonal components have values

$$W_i(t_1) = W_{1i}(t_1) + w_{2i}^2(t_1) \qquad (5.4.5)$$

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Thus, from the operator's viewpoint, the deterministic component contributes to the covariance of what the operator assumes is zero-mean white Gaussian noise. Thus, the Kalman filter and predictor of the model compute their estimates as if the disturbance were zero-mean white Gaussian noise with a covariance matrix $W(t)$ given by Eq (5.4.4). This approach results in an adaptive tracker (Refs 15, 34) since, as the acceleration of the target is increased, the noise disturbance will increase and thereby increase the gain of the Kalman filter. This results in more weight being given to recent observations, which is desirable when the target is maneuvering significantly.

Kleinman (Refs 20, 23) has modeled the plant disturbance without the white noise component; that is

$$w(t) = w_2(t)$$

where $w_2(t)$ is the deterministic disturbance discussed above. Again, as discussed above, the operator assumes this is a zero-mean white Gaussian noise. Kleinman chose to model the variance kernel of what the operator presumes to be zero-mean white Gaussian noise by

$$w_i(t)\delta(t) = 2 \tau_c w_{2i}^2(t)\delta(t)$$

where τ_c may be viewed as a correlation time for $w_2(t)$. The model equations which are developed below will assume the white noise component is present; however, we will compare results obtained using the two approaches in Chapter VII.

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As mentioned above, even though the operator (and therefore the model Kalman filter and predictor) assumes the disturbance is zero-mean white Gaussian noise, we will separate the deterministic and random effects of the disturbance in developing the propagation equations for the mean states and the corresponding covariances.

A modified augmented state equation is defined for the purpose of deriving these equations, i.e.,

$$\dot{x}_a(t) = A_a(t)x_a(t) + B_a U_c(t) + \Gamma_a w_a(t) \qquad (5.4.6)$$

where

$$x_a(t) = \begin{bmatrix} x(t) \\ ---- \\ u(t) \end{bmatrix}$$

$$A_a(t) = \begin{bmatrix} A(t) & \vdots & B(t) \\ ---------- \\ 0 & \vdots & -1/\tau_n \end{bmatrix}$$

$$B_a = \begin{bmatrix} 0 \\ ---- \\ 1/\tau_n \end{bmatrix}$$

$$\Gamma_a(t) = \begin{bmatrix} \Gamma(t) & \vdots & 0 \\ ---------- \\ 0 & \vdots & 1/\tau_n \end{bmatrix} \qquad \dot{E}\{w_a(t)\} = w_2(t)$$

$$w_a(t) = \begin{bmatrix} w_1(t) + w_2(t) \\ -------- \\ v_m(t) \end{bmatrix} \qquad E\{[w_a(t_1) - w_2(t_1)]$$

$$[w_a(t_2) - w_2(t_2)]^T\}$$

$$= \begin{bmatrix} W_1 & \vdots & 0 \\ ------ \\ 0 & \vdots & V_m \end{bmatrix} \delta(t_2 - t_1)$$

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Recall that $U_c(t)$ is the commanded control given by Eq (5.2.19).

Our objective is to derive expressions for propagating the mean states

$$x_{am}(t) = \dot{E}\{x_a(t)\} \qquad (5.4.7)$$

and covariance

$$P_a(t) = \dot{E}\{[x_a(t) - x_{am}(t)] [x_a(t) - x_{am}(t)]^T\} \qquad (5.4.8)$$

Consider first the covariance of the Kalman filter estimate of the delayed states. We will let $\sigma = t - \tau$ where τ is the perceptual delay time. The Kalman filter estimate is denoted by $\hat{x}_a(\sigma)$ and its covariance by

$$P_1(\sigma) = \dot{E}\{[x_a(\sigma) - \hat{x}_a(\sigma)] [x_a(\sigma) - \hat{x}_a(\sigma)]^T\} \qquad (5.4.9)$$

With the augmented state vector, the operator observation process is described by

$$y(\sigma) = C_a(\sigma)x_a(\sigma) + v(t) \qquad (5.4.10)$$

where

$$C_a(\sigma) = [C(t) \mid D(t)] \qquad (5.4.11)$$

and $v(t)$ is zero-mean white Gaussian noise with independent components and variance kernels

$$E\{v_i(t_1)v_i(t_2)\} = V_i(t_1)\delta(t_1 - t_2)$$

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Then the covariance of the Kalman filter estimate is propagated by

$$\dot{P}_1(\sigma) = A_a(\sigma)P_1(\sigma) + P_1(\sigma)A_a^T(\sigma) + \Gamma_a(\sigma)W(t)\Gamma_a^T(\sigma)$$

$$- P_1(\sigma)C_a^T(\sigma)V^{-1}(\sigma)C_a(\sigma)P_1(\sigma) \qquad (5.4.12)$$

Also, the estimate of the delayed states obeys

$$\dot{\hat{x}}(\sigma) = A_a(\sigma)\hat{x}_a(\sigma) + B_a U_c(\sigma)$$

$$+ P_1(\sigma)C_a^T(\sigma)V^{-1}(\sigma)\left[y(\sigma) - C_a(\sigma)\hat{x}_a(\sigma)\right]$$

$$(5.4.13)$$

Subtracting Eq (5.4.13) from Eq (5.4.6), we obtain the equation for propagating the estimation error of the delayed states, $e_1(\sigma) \overset{\Delta}{=} x_a(\sigma) - \hat{x}_a(\sigma)$

$$\dot{e}_1(\sigma) = A_a(\sigma)e_1(\sigma) - P_1(\sigma)C_a^T(\sigma)V^{-1}(\sigma)\left[C_a(\sigma)e_1(\sigma) + v(\sigma)\right]$$

$$+ \Gamma_a(\sigma)W(\sigma) \qquad (5.4.14)$$

Making the definitions

$$G(\sigma) \overset{\Delta}{=} P_1(\sigma)C_a^T(\sigma)V^{-1}(\sigma)$$

$$A_f(\sigma) \overset{\Delta}{=} A_a(\sigma) - G(\sigma)C_a(\sigma) \qquad (5.4.15)$$

where $G(\sigma)$ is the estimator gain and $A_f(\sigma)$ is the closed-loop filter gain. Eq (5.4.14) becomes

$$\dot{e}_1(\sigma) = A_f(\sigma)e_1(\sigma) - G(\sigma)v(\sigma) + \Gamma_a(\sigma)W(\sigma) \qquad (5.4.16)$$

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If we further define the closed loop control matrix by

$$A_c(\sigma) = A_a(\sigma) - B_a \lambda_c(\sigma)$$

where λ_{ci} is given by Eq (5.2.19) for $i = 1, 2, \cdots \ell + n$ and $\lambda_{c\,\ell+n+1} = 0$. Then the expression for propagating the estimate of the delayed states $\hat{x}_a(\sigma)$ in Eq (5.4.13) becomes

$$\dot{\hat{x}}_a(\sigma) = \left[A_a(\sigma) - B_a \lambda_c(\sigma) \right] \hat{x}_a(\sigma) + G(\sigma) \left[C_a(\sigma) e_1(\sigma) + v(\sigma) \right]$$

$$= A_c(\sigma) \hat{x}_a(\sigma) + G(\sigma) \left[C_a(\sigma) e_1(\sigma) + v(\sigma) \right] \qquad (5.4.17)$$

Now the predictor generates the best estimate $\hat{x}_a(t)$ from the Kalman filter output $\hat{x}_a(\sigma)$. This is obtained by evaluating

$$\hat{x}_a(t|\sigma) = E\{ x_a(t) \mid \hat{x}_a(\sigma) \} \qquad (5.4.18)$$

This leads to the following equation for propagating $\hat{x}_a(t|\sigma)$:

$$\dot{\hat{x}}_a(t|\sigma) = A_a(t) \hat{x}_a(t|\sigma) + B_a U_c(t)$$

$$\phi(t, t-\tau) G(\sigma) \left[C_a(\sigma) e_1(\sigma) + v(\sigma) \right]$$

$$(5.4.19)$$

Now the transition matrix $\phi(t, t-\tau)$ obeys

$$\frac{\partial \phi(t, t-\tau)}{\partial t} = A_a(t) \phi(t, t-\tau)$$

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For systems where $A_a(t)$ changes only slightly between $t-\tau$ and t (recall $\tau \approx .2$ sec), then

$$\phi(t, t-\tau) \approx e^{A_a(\sigma)\tau}$$

Also making the definition

$$K(\sigma) = e^{A_a(\sigma)\tau} G(\sigma)$$

Substituting these quantities into Eq (5.4.19) yields

$$\overset{\bullet}{\hat{x}}_a(t|\sigma) = A_c(t)\hat{x}_a(t|\sigma) + K(\sigma)\left[C_a(\sigma)e_1(\sigma) + v(\sigma)\right] \qquad (5.4.20)$$

The prediction error, $e_2(t)$, is the error introduced in estimating $\hat{x}_a(t|\sigma)$ given $\hat{x}_a(\sigma)$ and is obtained by

$$e_2(t) = \int_{t-\tau}^{t} e^{A_a(t-\varepsilon)} \Gamma_a(\varepsilon)w(\varepsilon)d\varepsilon \qquad (5.4.21)$$

Combining the above results yields the total state, $x_a(t)$, as

$$x_a(t) = \hat{x}_a(t/\sigma) + e^{A_a(\sigma)\tau}e_1(\sigma) + e_2(t) \qquad (5.4.22)$$

The mean state, $x_{am}(t)$, is determined by evaluating the expected value of the three terms of Eq (5.4.22); i.e.,

$$x_{am}(t) = E\{\hat{x}_a(t)\} + e^{A_a(\sigma)\tau} E\{e_1(\sigma)\} + E\{e_2(t)\}$$

$$(5.4.23)$$

From Eqs (5.4.20), (5.4.16), and (5.4.21) we can determine the following equations for propagating the expected values.

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$$\dot{\hat{x}}_{am}(t|\sigma) = A_c(t)\hat{x}_{am}(t|\sigma) + K(\sigma)C_a(\sigma)e_{1m}(\sigma) \qquad (5.4.24)$$

$$\dot{e}_{1m}(\sigma) = A_f(\sigma)e_{1m}(\sigma) + \left[\frac{\Gamma(\sigma)}{0}\right]w_2(\sigma) \qquad (5.4.25)$$

$$\hat{e}_{2m}(t) = \int_{t-\tau}^{t} e^{A_a(t-\varepsilon)}\left[\frac{\Gamma(\varepsilon)}{0}\right]w_2(\varepsilon)d\varepsilon \qquad (5.4.26)$$

where

$$\hat{x}_{am}(t|\sigma) = E\{\hat{x}_a(t|\sigma)\}$$

$$e_{1m}(\sigma) = E\{e_1(\sigma)\}$$

$$e_{2m}(t) = E\{e_2(t)\}$$

Note from the above coupled set of matrix equations that the _mean_ states occur because of the deterministic quantity, $w_2(t)$, in the disturbance.

Now the following definitions are established for calculating covariance and crosscovariance terms.

$$P_2(\sigma) \triangleq E\{[e_1(\sigma) - e_{1m}(\sigma)][e_1(\sigma) - e_{1m}(\sigma)]^T\}$$

$$P_3(\sigma) \triangleq E\{[\hat{x}_a(t) - \hat{x}_{am}(t)][e_1(\sigma) - e_{1m}(\sigma)]^T\}$$

$$P_4(t) \triangleq E\{[\hat{x}_a(t) - \hat{x}_{am}(t)][\hat{x}_a(t) - \hat{x}_{am}(t)]^T\} \qquad (5.4.27)$$

$$P_5(t) \triangleq E\{[e_2(t) - e_{2m}(t)][e_2(t) - e_{2m}(t)]^T\}$$

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From these definitions and the previous results, the following equations are obtained for propagating the above quantities:

$$\dot{P}_2(\sigma) = A_f(\sigma)P_2(\sigma) + P_2(\sigma)A_f^T(\sigma) + \Gamma_a(\sigma)\left[\begin{array}{c|c} W_1(\sigma) & 0 \\ \hline 0 & V_m(\sigma) \end{array}\right]\Gamma_a^T(\sigma)$$

$$+ G(\sigma)V(\sigma)G^T(\sigma) \tag{5.4.28}$$

$$\dot{P}_3(t) = A_c(t)P_3(t) + P_3(t)A_f^T(\sigma) + K(\sigma)C_a(\sigma)[P_2(\sigma) - P_1(\sigma)] \tag{5.4.29}$$

$$\dot{P}_4(t) = A_c(t)P_4(t) + P_4(t)A_c^T(t) + K(\sigma)C_a(\sigma)P_3^T(t)$$

$$+ P_3(t)C_a^T(\sigma)K^T(\sigma) + K(\sigma)V(\sigma)K^T(\sigma) \tag{5.4.30}$$

$$P_5(t) = \int_{t-\tau}^{t} e^{A_a(t-\varepsilon)}\Gamma_a(\varepsilon)\left[\begin{array}{c|c} W_1(\varepsilon) & 0 \\ \hline 0 & V_m(\varepsilon) \end{array}\right]\Gamma_a^T(\varepsilon)e^{A_a^T(t-\varepsilon)}d\varepsilon \tag{5.4.31}$$

Finally the covariance of the total states, $P_a(t)$, is determined by

$$P_a(t) = e^{A_a(\sigma)\tau}P_2(\sigma)e^{A_a^T(\sigma)\tau} + P_3(t)e^{A_a^T(\sigma)\tau}$$

$$+ e^{A_a(\sigma)\tau}P_3^T(t) + P_4(t) + P_5(t) \tag{5.4.32}$$

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The mean and covariance of the operator-observed data are

$$\dot{E}\{y(t)\} = y_m(t) = C_a(t)x_{am}(t) \qquad (5.4.33)$$

$$\dot{E}\{[y(t) - y_m(t)][y(t) - y_m(t)]^T\} = C_a(t)P_a(t)C_a^T(t)$$

$$(5.4.34)$$

In this chapter we have described the optimal control model for human response in some detail. The model has several parameters, the values of which influence the response of the system in simulating the human operator. These parameters are

τ_n = neuromotor delay (Eq (5.2.16))

τ = perception time delay (Eq (5.2.6))

ρ_i = proportionality constants for calculating observation noise variance kernels (Eq (5.2.8))

ρ_m = proportionality constant for calculating motor noise variance kernel (Eq (5.2.22))

W_1 = variance of white noise component of plant disturbance (Eq (5.4.2)).

These parameters appear in the coupled sets of matrix differential equations which propagate the mean states and covariances of the model response. This response would correspond to ensemble averaging many trials of a human operator in a tracking situation. The development has the

advantage that one can predict the mean response to deterministic target trajectories. In the next chapter we will describe a simulator system which was used to obtain data of operator response in tracking maneuvering targets flying deterministic trajectories. This will set the stage for a nontrivial parameter identification problem as discussed in Chapters II, III, and IV.

Chapter VI

SIMULATOR SYSTEM DYNAMICS, STATE SPACE
MODELING, AND RESULTS OF EXPERIMENTAL TRIALS

The purpose of this chapter is to describe the simulator system which was used to obtain experimental data with operators performing a tracking task. The simulator dynamics will be described and the state space equations will be developed to correspond with the development of the previous chapter. In addition, visual thresholds will be discussed in the context of the simulator display, as well as an indifference threshold effect due to the finite size of the displayed target. The deterministic target trajectories which were used for the simulator trials and the data resulting from ensemble averaging many trial runs will be presented in this chapter as well.

6.1 General Description of Simulator

The simulator used in this research is maintained and operated by the Aerospace Medical Research Laboratory, Wright-Patterson AFB, Ohio. Only the portions of the system pertinent to the discussion will be presented.

The system uses two people in the tracking scheme. Each person has an independent set of controls for tracking one axis (azimuth or elevation). The operators receive tracking information by way of a television monitor. A large screen presents data only for the first ten seconds

to allow acquisition. After that both operators receive data from a small 4.5-inch by 4.5-inch monitor. The sighting reticle consists of pairs of narrow vertical and horizontal lines with a spacing of about .002 inches. Each operator is provided with handwheels to position the geometrical center of the TV camera which provides the visual signal to the display monitor. The elevation operator is attempting to maintain the horizontal lines on the target and the azimuth operator controls the vertical pair of lines. The operators have the choice of either position control or a rate-aided mode of control (see Section 6.2 on simulator dynamics). Only one mode select switch is provided so that each axis is always tracked in the same mode. The usual operation is to acquire the target using the position control followed by operation in the rate-aided mode while tracking.

6.2 Simulator Dynamics

The geometry of the tracking task is shown in Fig. 6.1. From this figure we see that

Θ_A = azimuth angle of target

Θ_E = elevation angle of target

Θ_{SA} = azimuth angle of sight/camera

Θ_{SE} = elevation angle of sight/camera

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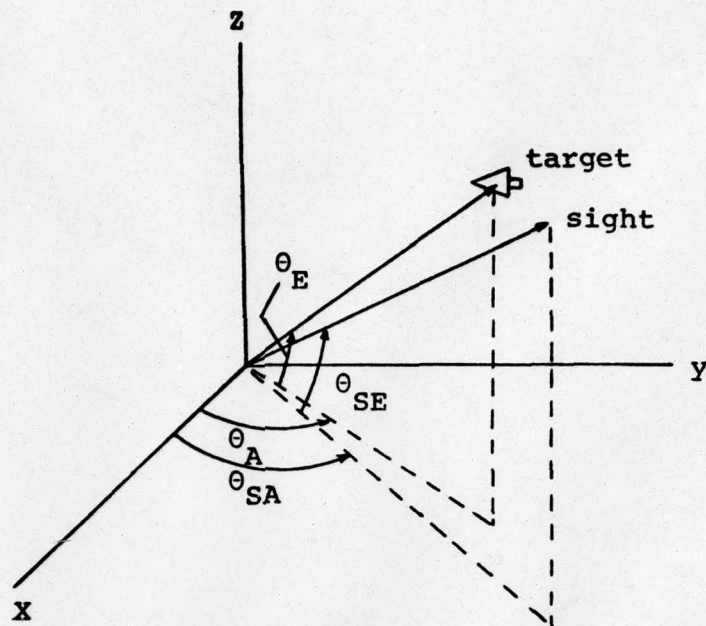
Fig. 6.1   Tracking Geometry

The operators view the traverse error (in the slant plane) and the elevation error on the monitor; therefore we make the following additional definitions:

θ_T = traverse angle of target in slant plane

θ_{ST} = traverse angle of sight/camera

so that

$$\theta_T = \theta_A \cos\theta_E \qquad\qquad (6.2.1)$$

$$\theta_{ST} = \theta_{SA} \cos\theta_E$$

and

Traverse error = $\Theta_T - \Theta_{ST}$

Elevation error = $\Theta_E - \Theta_{SE}$

(6.2.2)

The dynamics for the two axes are shown in Fig. 6.2.
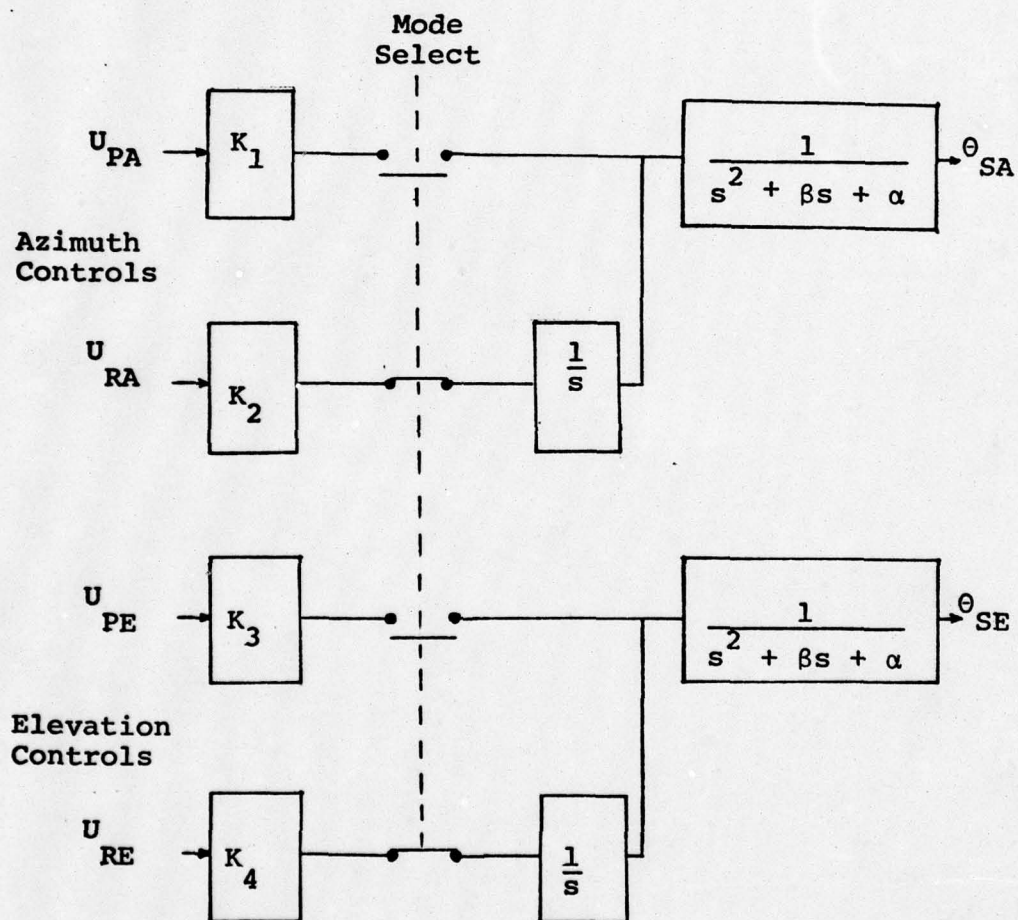
Fig. 6.2   Two Operator Simulator Tracking Loop Dynamics

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From Fig. 6.2, the following transfer functions are obtained:

For position control

$$\frac{\theta_{SA}}{U_{PA}} = \frac{K_1}{s^2 + \beta s + \alpha} \qquad (6.2.3)$$

$$\frac{\theta_{SE}}{U_{PE}} = \frac{K_3}{s^2 + \beta s + \alpha} \qquad (6.2.4)$$

For rate control

$$\frac{\theta_{SA}}{U_{RA}} = \frac{K_1}{s(s^2 + \beta s + \alpha)} \qquad (6.2.5)$$

and

$$\frac{\theta_{SE}}{U_{RE}} = \frac{K_4}{s(s^2 + \beta s + \alpha)} \qquad (6.2.6)$$

The following values for the transfer function constants correspond to those of the simulator used for obtaining experimental data:

$\alpha = 64$	$K_1 = 3.2$	$K_2 = 3.156$
$\beta = 12$	$K_3 = .8$	$K_4 = 8.533$

6.3 Simulator State Space Equations

The state space equations will be developed to correspond with the optimal control model presented in Chapter V.

The states of the disturbance noise and system dynamics obey the equation

$$\dot{x}(t) = Ax(t) + \Gamma w(t) + B(t)u(t) \qquad (6.3.1)$$

where, assuming rate-aided control,

$$x(t) = \begin{bmatrix} \dot{\Theta}_T \\ \dot{\Theta}_{ST} \\ \ddot{\Theta}_{ST} \\ \Theta_{ST} - \Theta_T \\ \dot{\Theta}_E \\ \dot{\Theta}_{SE} \\ \ddot{\Theta}_{SE} \\ \Theta_{SE} - \Theta_E \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & -\beta & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & -\beta & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

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$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$w(t) = w_1(t) + w_2(t)$$

$$B(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ K_2 \cos\theta_E(t) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & K_y \\ 0 & 0 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} U_{Ra}(t) \\ U_{RE}(t) \end{bmatrix}$$

As discussed in Section 5.4, $w_1(t)$ is zero-mean white Gaussian noise and $w_2(t)$ is a deterministic disturbance. We will use target angular acceleration as the deterministic portion of the disturbance so that

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$$w_2(t) = \begin{bmatrix} \ddot{\Theta}_T(t) \\ \ddot{\Theta}_E(t) \end{bmatrix}$$

It is noted that the equations for the two axes can be essentially decoupled. That is for the azimuth axis the state equation is

$$\dot{x}(t) = A\, x(t) + \Gamma\, w(t) + B(t)u(t) \qquad (6.3.1)$$

where

$$x(t) = \begin{bmatrix} \dot{\Theta}_T \\ \dot{\Theta}_{ST} \\ \ddot{\Theta}_{ST} \\ \Theta_{ST} - \Theta_T \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ K_2 \cos\Theta_E(t) \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha & -\beta & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \qquad u(t) = U_{RA}(t)$$

$$\Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$w(t) = w_1(t) + \ddot{\Theta}_T(t)$$

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Similarly, the elevation axis can be described by an equation of the form of Eq (6.3.1), with

$$x(t) = \begin{bmatrix} \dot{\theta}_E \\ \dot{\theta}_{SE} \\ \ddot{\theta}_{SE} \\ \theta_{SE} - \theta_E \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ K_4 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha & -\beta & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \qquad u(t) = U_{RE}(t)$$

$$w(t) = w_1(t) + \ddot{\theta}_E(t)$$

Note that B is not a function of time here. The state vector in each of these decoupled equations has the first element as state for the noise model and the remaining three are the states of the system dynamics (see Eq (5.2.3)).

The variables for each axis displayed to the operators are

$$y_{\tilde{d}}(t) = C\,x(t) \qquad\qquad (6.3.2)$$

where

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

That is, each operator is assumed to observe angular error and angular error rate. As discussed in Chapter V, this is

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consistent with the usual assumption that if a variable is displayed visually to the operator, then the rate of change of that quantity is also discerned.

Each operator wants to minimize his own cost functional, i.e., for azimuth

$$J = \lim_{t_f \to \infty} E \left\{ \frac{1}{t_f} \int_0^{t_f} \left[x^T(t) Q \, x(t) + g(t) \dot{u}_{RA}^2(t) \right] dt \right\}$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

The elevation operator has a similar cost functional.

6.4 Simulator Riccati Equation (see Section 5.3)

The remaining discussion will be confined to the azimuth axis, with obvious extension to the elevation axis. We will approach the solution to the Riccati equation for the simulator as discussed in Section 5.3. An augmented state vector is defined as in Eq (5.2.11)

$$x_R(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{6.4.1}$$

with

$$\dot{x}_R(t) = A_R(t) x_R(t) + B_R \dot{u}(t) \tag{6.4.2}$$

where

$$A_R(t) = \begin{bmatrix} A & B(t) \\ 0 & 0 \end{bmatrix}$$

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and

$$B_R = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that the subscript R is used to denote the augmented state equation leading to the Riccati equation as distinguished from another augmented state equation in Section 6.5. The optimal gains, λ, are obtained by

$$\lambda(t) = \frac{1}{g} B_R^T P_R(t) \tag{6.4.3}$$

Recall that $P_R(t)$ is the solution of the Riccati equation at time t with $t_f \to \infty$; i.e., the quasi-static steady state solution to

$$P_C(t)A_R(t) + A_R^T(t)P_C(t) - P_C(t)B_R \frac{1}{g(t)} B_R^T P_R(t) + Q_R = 0 \tag{6.4.4}$$

where

$$\overrightarrow{Q_R} = \begin{bmatrix} Q & \vdots & 0 \\ \text{---} & \text{---} & \text{---} \\ 0 & \vdots & 0 \end{bmatrix}$$

Partitioning the Riccati equation as shown in Eq (5.3.1) leads to the following definitions for the simulator case:

$$A_n = 0 \qquad\qquad A_b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

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$$A_d = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & -\beta & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B_C(t) = \begin{bmatrix} 0 \\ K_2 \cos\theta_E(t) \\ 0 \end{bmatrix}$$

Note that time variations occur in the azimuth case due to the crosscoupling of the elevation angle (appearing in $B_C(t)$) that do not occur in the elevation axis. Then the equations corresponding to Eqs (5.3.2) through (5.3.7) become

$$P_{12} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} P_{12}^T - \frac{1}{g} P_{13} P_{13}^T = 0 \qquad (6.4.5)$$

$$P_{12} \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & -\beta & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} P_{22} - \frac{1}{g} P_{13} P_{23}^T = 0 \qquad (6.4.6)$$

$$P_{12} \begin{bmatrix} 0 \\ K_2 \cos\theta_E(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} P_{23} - \frac{1}{g} P_{13} P_{33} = 0 \qquad (6.4.7)$$

$$P_{22} \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & -\beta & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha & 1 \\ 1 & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix} P_{22} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{bmatrix}$$

$$- \frac{1}{g} P_{23} P_{23}^T = 0 \qquad (6.4.8)$$

$$P_{22} \begin{bmatrix} 0 \\ K_2 \cos\theta(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha & 1 \\ 1 & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix} P_{23} - \frac{1}{g} P_{23} P_{33} = 0$$

$$(6.4.9)$$

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$$2P_{23}^T \begin{bmatrix} 0 \\ K_2 \cos\theta \ (t) \\ 0 \end{bmatrix} - \frac{1}{g} P_{33}^2 = 0 \tag{6.4.10}$$

Also recall from Section 5.3 that if $\tau_n = g(t)/P_{33}(t)$ is held constant, this completely determines the gains, $\lambda(t)$, regardless of the value of q. The procedure used here was to specify a value of τ_n, set q = 1, and then adjust g(t) to obtain the desired value of $\tau_n = g(t)/P_{33}(t)$. (A value within ± .001 of the desired value was considered adequate.) With q and g(t) specified, Eqs (6.4.5) through (6.4.10) can be solved for the required unknowns to obtain the feedback gains $\lambda(t)$. The details of a method for obtaining a solution are discussed in Appendix E. As shown in Eq (5.3.9), $\lambda(t)$ is determined by

$$\lambda(t) = \frac{1}{g(t)} \begin{bmatrix} P_{13}^T & | & P_{23} & | & P_{33} \end{bmatrix} \tag{6.4.11}$$

6.5 Simulator Mean State and Covariance Propagation (see Section 5.4)

The augmented state Eq (5.4.6) takes the following form for the simulator azimuth axis:

$$\dot{x}_a(t) = A_a(t)x_a(t) + B_a U_c(t) + \Gamma_a w_a(t) \tag{6.5.1}$$

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where

$$x_a(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \dot{\Theta}_T(t) \\ \dot{\Theta}_{ST}(t) \\ \ddot{\Theta}_{ST}(t) \\ \Theta_{ST}(t) - \Theta_T(t) \\ U_{RA}(t) \end{bmatrix}$$

$$A_a(t) = \begin{bmatrix} A & B(t) \\ 0 & -1/\tau_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\alpha & -\beta & 0 & K_2 \cos\theta_E(t) \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\tau_n \end{bmatrix}$$

$$B_a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1/\tau_n \end{bmatrix}$$

$$\Gamma_a = \begin{bmatrix} \Gamma & 0 \\ 0 & 1/\tau_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1/\tau_n \end{bmatrix}$$

$$w_a(t) = \begin{bmatrix} w_1(t) + \ddot{\Theta}_T(t) \\ \hline v_m(t) \end{bmatrix}$$

and

$$U_c(t) = -\lambda_c(t)\hat{x}(t)$$

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with

$$\lambda_{ci}(t) = \tau_n \, \lambda_i(t) \qquad i = 1, 2, 3, 4$$

With the above, the following equations result for propagating the mean states as derived in Section 5.4:

$$x_{am}(t) = \hat{x}_{am}(t|\sigma) + e^{A_a(\sigma)} e_{1m}(\sigma) + e_{2m}(t) \qquad (6.5.2)$$

where $x_{am}(t) = E\{x_a(t)\}$ and $\sigma = t - \tau$.

The equations for propagating $\hat{x}_{am}(t)$, $e_{1m}(\sigma)$, and $e_{2m}(t)$ are expressed in Eqs (5.4.24) through (5.4.26).

The covariance

$$P_a(t) = E\{\left[x_a - x_{am}(t)\right]\left[x_a(t) - x_{am}(t)\right]^T\} \qquad (6.5.3)$$

is determined from

$$P_a(t) = e^{A_a(\sigma)\tau} P_2(\sigma) e^{A_a^T(\sigma)\tau} + P_3(t) e^{A_a^T(\sigma)\tau}$$

$$+ e^{A_a(\sigma)\tau} P_3^T(t) + P_4(t) + P_5(t) \qquad (6.5.4)$$

where $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$, and $P_5(t)$ are propagated by Eqs (5.4.12) and (5.4.28) through (5.4.31).

6.6 Threshold Effects

As previously discussed, the simulator operators observe

$$y(t) = \begin{bmatrix} \Theta_{ST} - \Theta_T \\ \dot{\Theta}_{ST} - \dot{\Theta}_T \end{bmatrix}$$

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in the traverse plane and corresponding information in the elevation plane. When the magnitude of the angle error and angle error rates are below certain threshold values, the operator will not detect any change. Typical values for these visual thresholds in terms of arc angles from the operator's eyes are (Ref 21) .05 degrees for angular position and .18 degrees/sec for angular rate. Considering the field of view for the monitor, the monitor dimensions and the distance of the operator's eye from the monitor, one arrives at a display gain of 5. Therefore the visual thresholds after accounting for display gain are .01 degrees $(.175 \times 10^{-3}$ rad) in position and .036 degrees/sec $(.628 \times 10^{-3}$ rad/sec) for rate.

Another threshold effect apparently takes place in conjunction with this simulator. The target covers a sizeable portion of the screen, especially at short ranges. Discussions with the operators revealed that they did not try to improve the position of the cross hairs when they were within a few feet of the apparent spatial centroid. The target image was an F-4, which is 58 feet long, 16 feet high, and has a 38 1/2 foot wing span. The operators' indifference was probably due to (1) tracking within a few feet of the aircraft centroid was well within the vulnerable area which would be scored a hit, and (2) the change in shape of the aircraft image due to pitch, roll, and yaw relative to the tracking site tends to smooth out the operator's response when sufficiently close to the desired

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nominal position. A survey was taken of seven tracking
teams (14 people)[1] and it was found that the average region
of indifference was about ±8 feet in the traverse plane and
±5 feet in the elevation plane. Thus a second threshold is
introduced which is a function of target range; i.e., the
magnitude of the threshold increases as the size of the
image on the screen increases. Incorporation of this effect
is considered a significant refinement since this same
apparent phenomena has been observed in other experimental
data, but not adequately explained. In view of this, a
threshold was incorporated as follows:

Position Threshold =

$$\text{Max}\left[.175 \times 10^{-3} \text{ rad}, \frac{\Delta x}{\text{Range}}\right]$$

Rate Threshold =

$$\text{Max}\left[.628 \times 10^{-3} \text{ rad}, .628 \times 10^{-3}\left[\frac{\text{Pos. Thres.}}{.175 \times 10^{-3}}\right]\right]$$

where

Δx = 8 feet for traverse axis

Δx = 5 feet for elevation axis

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[1]Eight teams of two people each were used in the simu-
lation runs; one team was not available for this survey.

The threshold effect is incorporated into the model as in Ref 2. Each perceived output is modified according to

$$y_{pi}(t) = f_i(y_i(t)) + v_{yi}(t) \tag{6.6.1}$$

where the threshold nonlinear function is

$$f(x) = \begin{cases} x - a_i & x \geq a_i \\ 0 & -a_i < x < a_i \\ x + a_i & x \leq -a_i \end{cases} \tag{6.6.2}$$

To facilitate accounting for this nonlinear effect in the propagation equations, we want to find a function $\hat{f}(y)$ such that the difference

$$d(t) = f(y(t)) - \hat{f}(y) \cdot y(t) \tag{6.6.3}$$

is minimized in the mean squared statistical sense. When $y(t)$ is assumed to be a zero-mean Gaussian random variable with variance σ, it has been shown (Ref 21) that

$$\hat{f}(y) = \text{erfc}\left(\frac{a}{\sigma \sqrt{2}}\right) \tag{6.6.4}$$

where

$$\text{erfc}(b) = 1. - \frac{2}{\sqrt{\pi}} \int_0^b \rho^{-w^2} dw.$$

Thus it is assumed that the operator perceives

$$y(t) = \hat{f}(y)y(t) + v(t) \tag{6.6.5}$$

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Recall from Eq (5.4.10) that

$$y(t) = C_a x_a(t) + v(t) \qquad (6.6.6)$$

where $C_a = \begin{bmatrix} C & \vdots & D \end{bmatrix}$

so that, considering threshold effects

$$y(t) = \hat{f}(y) C_a x_a(t) + v(t) \qquad (6.6.7)$$

Now define $C' = \hat{f}(y) C_a$, to obtain

$$y(t) = C' x_a(t) + v(t) \qquad (6.6.8)$$

Recall that $v(t)$ is zero-mean white Gaussian noise with independent components and variance kernels $V_i(t)$. Since only the quantity $C_a^T V^{-1}(t) C_a$ appears in the model equations, an equivalent result is obtained by assuming that the perceived data is

$$y(t) = y_d(t) + v(t) \hat{f}^{-1}(y)$$

$$= C_a x_a(t) + v(t) \hat{f}^{-1}(y) \qquad (6.6.9)$$

The observation noise covariance without threshold effects is obtained by

$$V_i = \rho_i \, \text{var}(y) \qquad (6.6.10)$$

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Considering threshold effects, the covariance is obtained by

$$V_i = \left[\rho_i \ \text{var}(y)\right] \cdot \hat{f}^{-2}(y) \qquad (6.6.11)$$

where

$$f(y) = \text{erfc}\left(\frac{a}{\sigma_y \sqrt{2}}\right)$$

6.7 Target Trajectories

Two target trajectories are used to generate simulator data. Trajectory 1 is a fly-by (see Fig. 6.7.1). The trajectory initiates at $x_0 = 10000$ feet, $y_0 = 5000$ feet, and $z_0 = 5000$ feet and flies straight and level as shown at a velocity of 500 feet/sec. Trajectory 2 is shown in Fig. 6.7.2 with coordinates given in Table VI-I. Equations used to calculate angular position, velocity, and acceleration for the fly-by trajectory are:

$$\theta_A = \tan^{-1}\left[\frac{x_0 - v_0 t}{y_0}\right] \qquad (6.7.1)$$

$$\theta_T = \theta_A \cos\theta_E = \tan^{-1}\left[\frac{x_0 - v_0 t}{y_0}\right] \cos\theta_E \qquad (6.7.2)$$

$$\dot{\theta}_T = \dot{\theta}_A \cos\theta_E - \theta_A \dot{\theta}_E \sin\theta_E \qquad (6.7.3)$$

$$\ddot{\theta}_T = \ddot{\theta}_A \cos\theta_E - \dot{\theta}_A \dot{\theta}_E \sin\theta_E - \dot{\theta}_A \dot{\theta}_E \sin\theta_E$$

$$- \theta_A \ddot{\theta}_E \sin\theta_E - \theta_A \dot{\theta}_E^2 \cos\theta_E \qquad (6.7.4)$$

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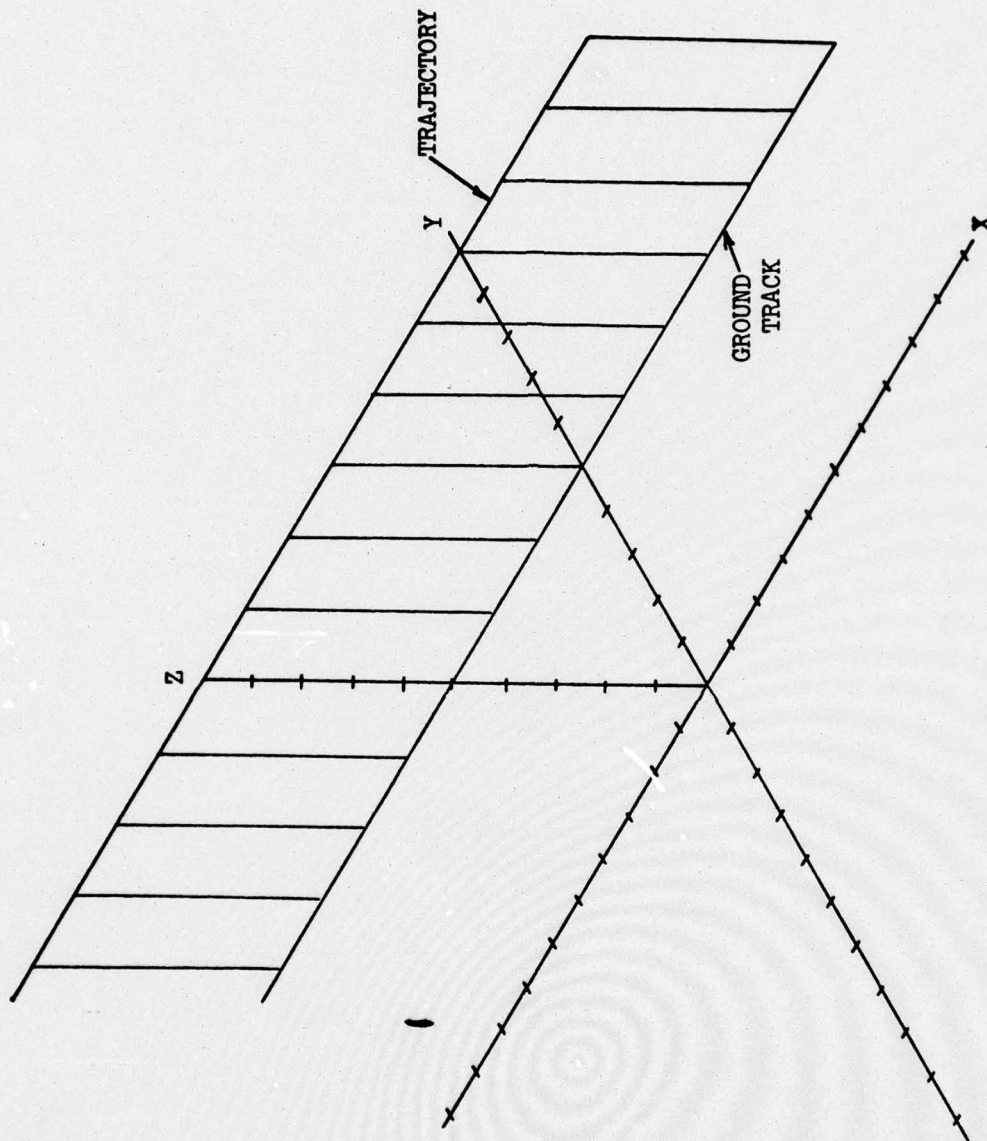
FIG. 6.7.1
TRAJECTORY PLOT (TRAJECTORY 1)
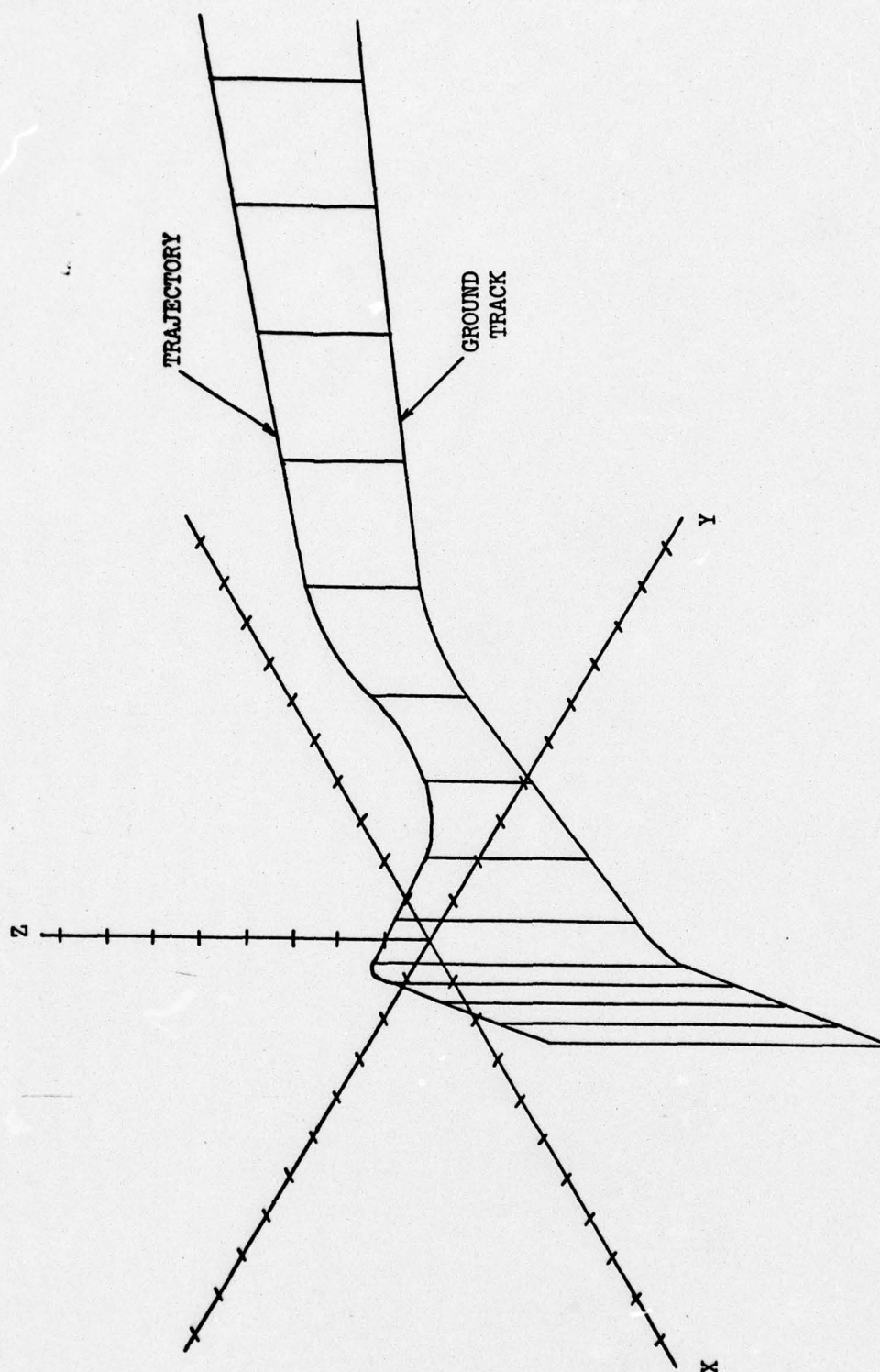
FIG. 6.7.2
TRAJECTORY PLOT (TRAJECTORY 2)

Table 6.1

TIME VS. POSITION (TRAJECTORY 2)

Time	- x	y	z
.00	-11318	8723	7441
.50	-11129	8389	7441
1.00	-10900	8455	7441
1.50	-10691	8321	7441
2.00	-10482	8187	7441
2.50	-10273	8053	7441
3.00	-10064	7919	7441
3.50	-9855	7785	7441
4.00	-9645	7651	7441
4.50	-9437	7517	7441
5.00	-9228	7383	7441
5.50	-9219	7249	7441
6.00	-8810	7115	7441
6.50	-8601	6981	7441
7.00	-8392	6847	7441
7.50	-8183	6713	7441
8.00	-7974	6579	7441
8.50	-7763	6445	7441
9.00	-7538	6311	7441
9.50	-7347	6177	7441
10.00	-7133	6043	7441
10.50	-6929	5909	7427
11.00	-6720	5775	7378
11.50	-6511	5641	7285
12.00	-6302	5507	7149
12.50	-6093	5373	6969
13.00	-5884	5239	6745
13.50	-5675	5103	6544
14.00	-5452	4098	6336
14.50	-5219	4911	6122
15.00	-4980	4839	5902
15.50	-4736	4778	5676
16.00	-4487	4726	5445
16.50	-4235	4879	5210
17.00	-3978	4837	4972
17.50	-3719	4598	4729
18.00	-3455	4561	4483
18.50	-3189	4526	4234
19.00	-2919	4490	3981
19.50	-2644	4458	3727
20.00	-2361	4423	3478
20.50	-2065	4388	3236
21.00	-1752	4350	3013
21.50	-1425	4311	2811
22.00	-1084	4270	2630
22.50	-732	4228	2471

Table 6.1   (continued)

Time	- x	y	z
23.00	-370	4184	2336
23.50	-1	4140	2223
24.00	374	4095	2135
24.50	752	4050	2070
25.00	1132	4004	2030
25.50	1511	3059	2014
26.00	1888	3014	2022
26.50	2260	3869	2053
27.00	2627	3829	2110
27.50	2990	3812	2171
28.00	3331	3826	2242
28.50	3706	3873	2302
29.00	4064	3951	2554
29.50	4392	4039	2400
30.00	4713	4198	2441
30.50	5024	4364	2477
31.00	5313	4530	2509
31.50	5802	4752	2541
32.00	5891	4946	2573
32.50	6180	5140	2625
33.00	6469	5334	2637
33.50	6758	5328	2659
34.00	7047	5722	2701
34.50	7336	5816	2733
35.00	7625	6110	2765
35.50	7914	6304	2797
36.00	8203	6498	2829
36.50	8492	6692	2861
37.00	8781	6886	2893
37.50	9070	7280	2926
38.00	9359	7274	2957
38.50	9648	7468	2939
39.00	9937	7662	3021
39.50	10226	7856	3053
40.00	10515	8050	3085
40.50	10804	8244	3117
41.00	11093	8438	3149
41.50	11382	8532	3181
42.00	11671	8826	3213
42.50	11960	9020	3245
43.00	12249	9214	3277
43.50	12538	9408	3309
44.00	12827	9602	3341
44.50	13116	9796	3373
45.00	13405	9990	3405

$$\dot{\theta}_A = \frac{-v_o}{y_o} \cos^2 \theta_A \qquad (6.7.5)$$

$$\ddot{\theta}_A = \frac{v_o}{y_o} \theta_A \sin 2\theta_A \qquad (6.7.6)$$

$$\theta_E = \tan^{-1} \left[\frac{z_o}{x^2 + y_o^2} \right] \qquad (6.7.7)$$

$$\dot{\theta}_E = \left[\frac{v_o \, y_o}{z_o^2} \right] \sin^2 \theta_E \, \tan\theta_E \, \tan\theta_A \qquad (6.7.8)$$

$$\ddot{\theta}_E = \dot{\theta}_E^2 \, \tan\theta_E + \frac{3}{\tan\theta_E} - \tan\theta_E \left[\frac{v_o}{z_o} \sin\theta_E \right]^2 \qquad (6.7.9)$$

For the second trajectory, angular position at time t is calculated by linearly interpolating between data points given in Table VI-I in increments of .05 seconds (the data displayed to the operators was based on a linear inter-polation between these points also). The angular velocity and acceleration were obtained by fitting seven points (three points on either side of the desired point plus the point itself) of the angular position to a polynomial

$$\theta = K_o + K_1 \, t + K_2 \, t^2 \qquad (6.7.10)$$

and then the velocity and acceleration are obtained from

$$\dot{\theta} = K_1 + 2 K_2 \, t \qquad (6.7.11)$$

and

$$\ddot{\theta} = 2 K_2 \qquad (6.7.12)$$

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6.8 Simulator Data

Eight teams of two people each were used in the simulation runs. The tracking error data was recorded as a function of time on magnetic tape for many runs of each team for both of the trajectories. Ensemble averages were computed at .2 second intervals over 56 runs for each trajectory. In other words, at each time point, the mean tracking error was computed for each axis as follows

$$\Theta_{sm}(t_j) = \frac{1}{N} \sum_{i=1}^{N} \Theta_{si}(t_j) \qquad (6.8.1)$$

where

$\Theta_{sm}(t_j)$ = mean tracking error at time t_j

$\Theta_{si}(t_j)$ = tracking error at time t_j for run i

N = total number of runs = 56

The standard deviation was computed using the relation

$$\sigma_s(t_j) = \left[\frac{1}{N} \sum_{i=1}^{N} \Theta_{si}^2(t_j) - \Theta_{sm}^2(t_j) \right]^{\frac{1}{2}} \qquad (6.8.2)$$

where

$\sigma_s(t_j)$ = standard deviation at time t_j

No attempt was made to subgroup data by teams since the objective here was to obtain a large number of trials over a crosssection of teams.

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Plots of this data for each axis for the two trajectories are shown in Figs. 6.8.1, 6.8.2, 6.8.3, and 6.8.4. The large errors initially occur during acquisition of the target by the trackers. On each of the figures we have denoted some important time points associated with the trajectories. For example, for the fly-by of Trajectory 1, the time of closest approach (crossover) is indicated. Note that the point of crossover is not so close that track is lost. For Trajectory 2, the times of azimuth crossover and minimum altitude are indicated.

In the next chapter we will use this data in conjunction with a gradient computational procedure to evaluate the optimal control model parameters.

FIG. 6.8.1
AZIMUTH TRACKING ERROR
TRAJECTORY ONE
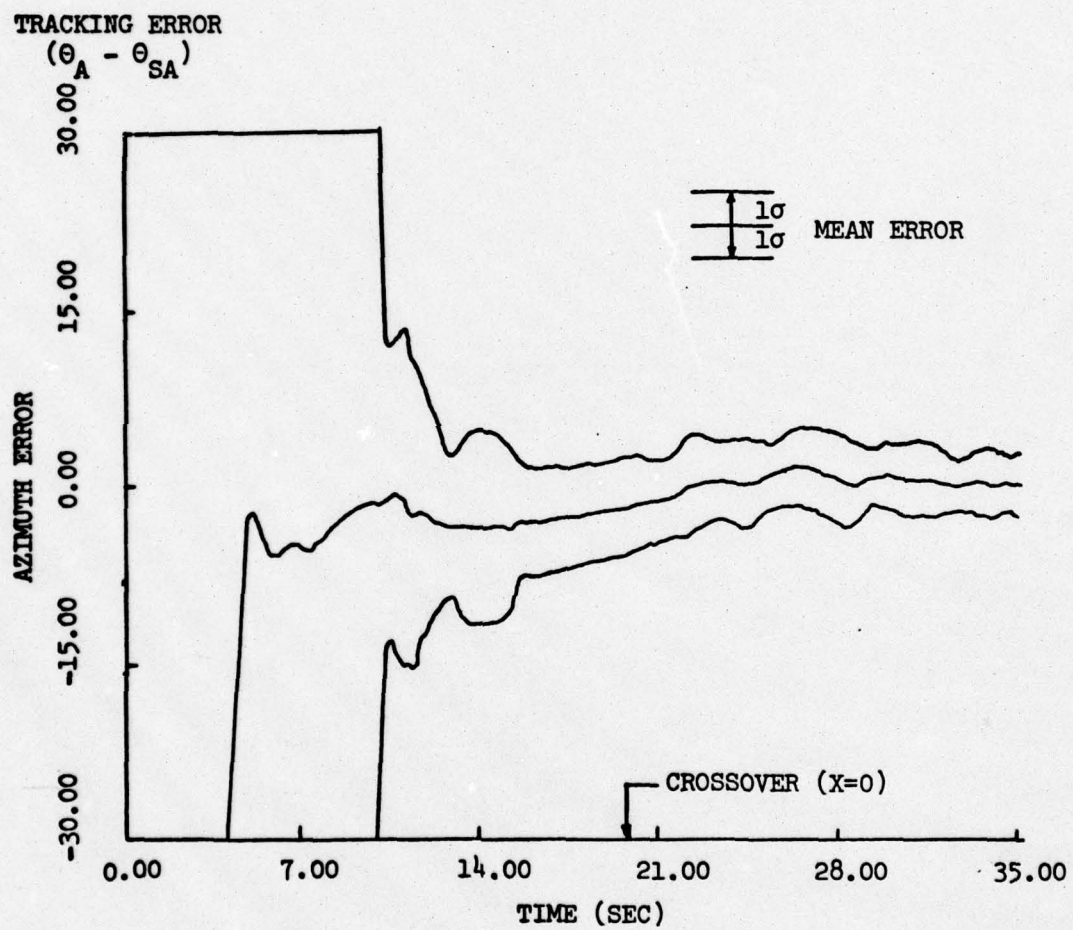
FIG. 6.8.2
ELEVATION TRACKING ERROR
TRAJECTORY ONE

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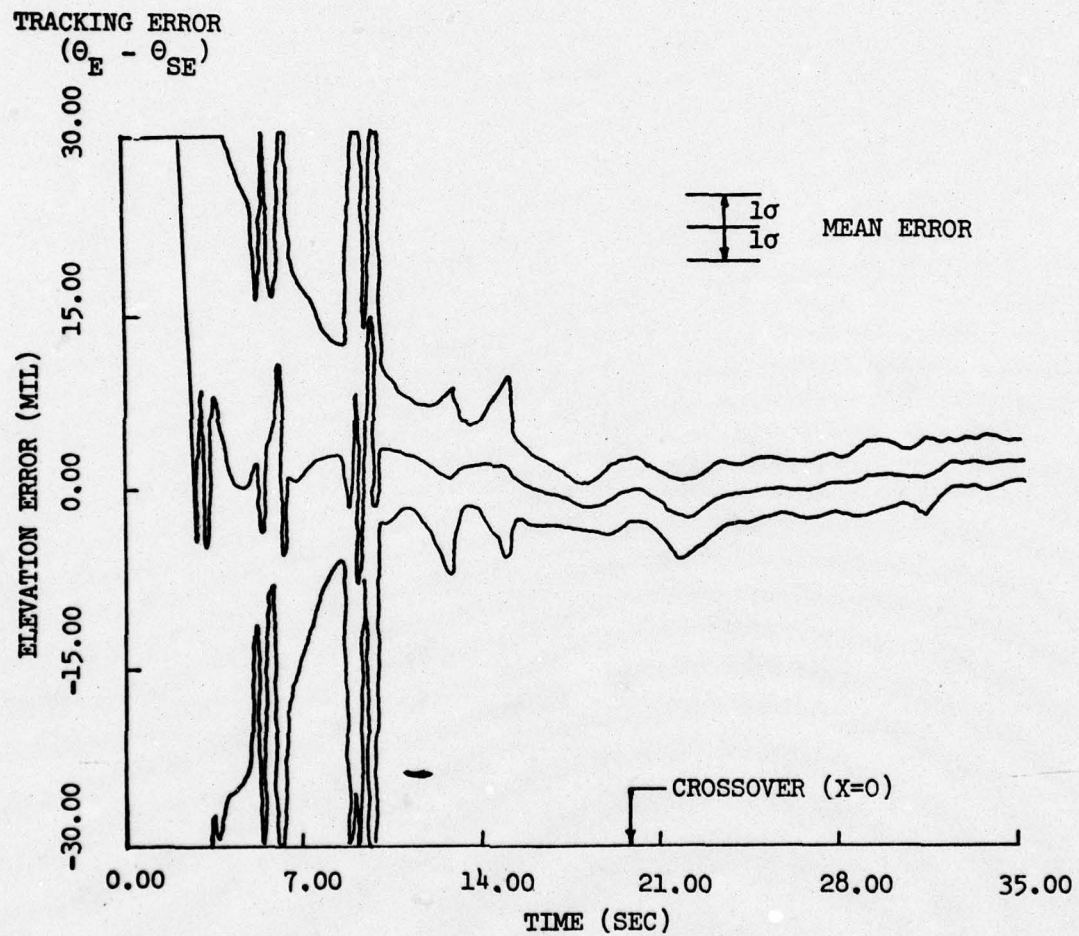
FIG. 6.8.3
AZIMUTH TRACKING ERROR
TRAJECTORY 2

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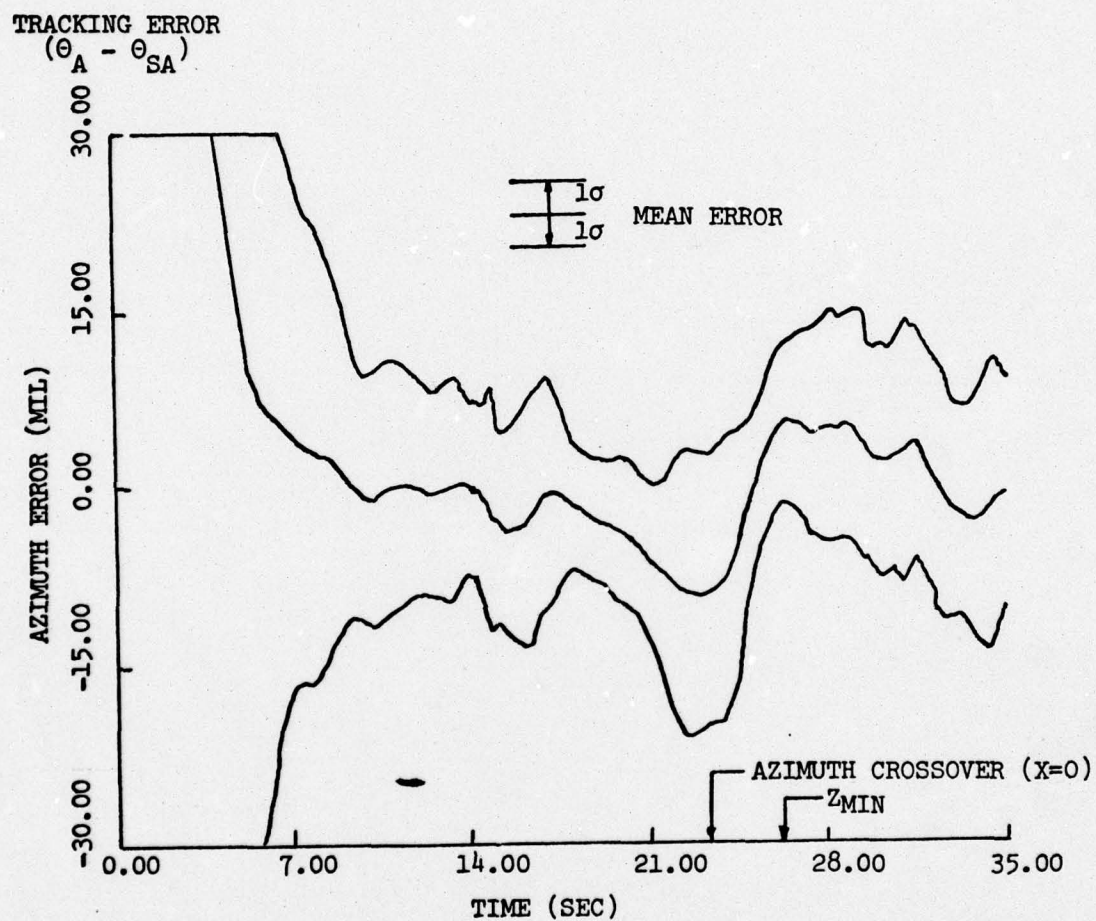
FIG. 6.8.4
ELEVATION TRACKING ERROR
TRAJECTORY 2

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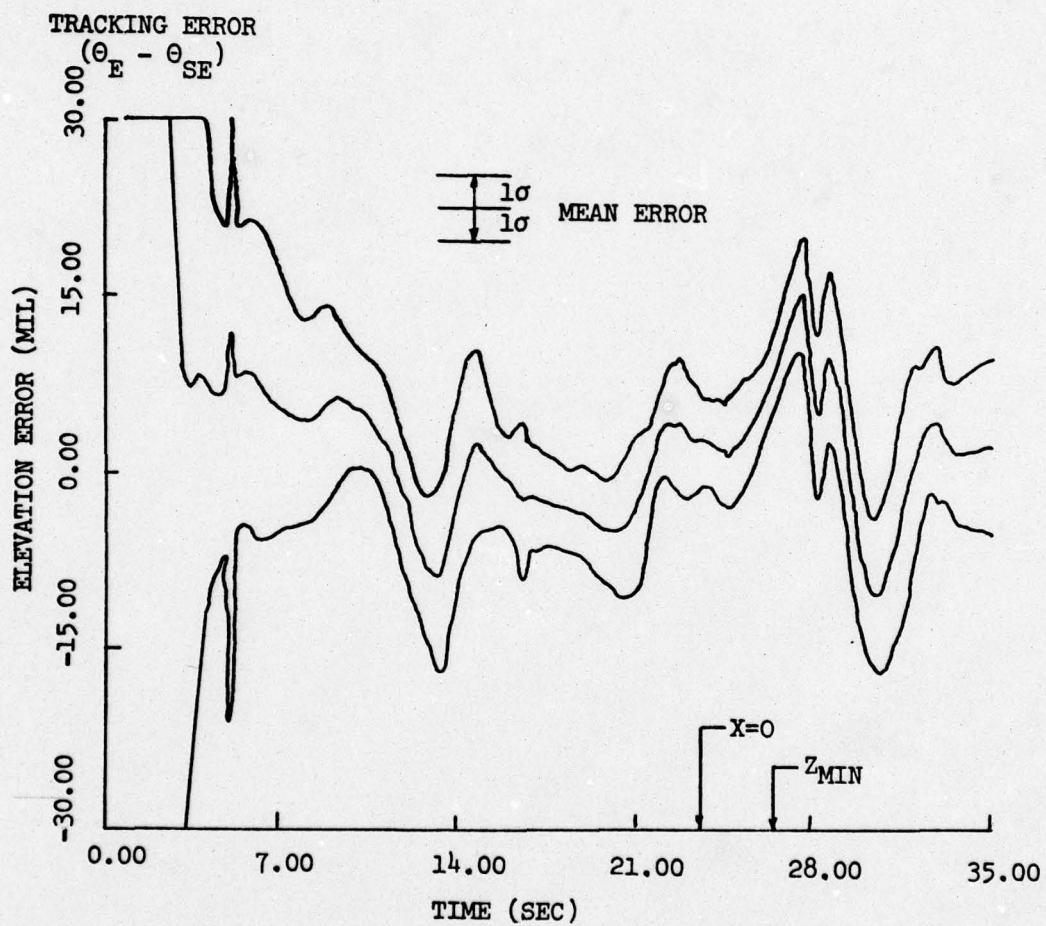
Chapter VII

IDENTIFIABILITY OF OPTIMAL CONTROL MODEL AND COMPUTATIONAL RESULTS

In this chapter, the identifiability of the optimal control model parameters will be evaluated, and the results of computations to estimate the values of the parameters will be presented. With the structure of the model for human performance assumed to be that presented in Chapters V and VI, there are several parameters whose values need to be estimated based on measured data. The parameters of interest are:

τ_n--neuromotor delay time

τ --perception time delay

ρ_i--proportionality constants for calculating observation noise variance kernels

ρ_m--proportionality constant for calculating motor noise variance kernel

W_1--variance of white noise component of scalar plant disturbance.

These parameters are viewed as elements of a vector ϕ; i.e.,

$$\phi = \begin{bmatrix} \tau_n & \tau & \rho_1 & \rho_2 & \rho_m & W_1 \end{bmatrix}^T \tag{7.1}$$

Recall from Chapter V that

$x_{am}(t)$ = mean states of optimal control model

$P_a(t)$ = covariances of states of optimal control model

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As discussed previously, one can construct a set of measured data by computing ensemble averages and variances of many trials on a simulator. An example of this type of data is shown in Figs. 6.8.1 through 6.8.4. Suppose we define a vector $P_{axx}(t)$, which is formed from the diagonal components of $P_a(t)$. With this definition we can form a state vector

$$x(t) = \begin{bmatrix} x_{am}(t) \\ \sqrt{P_{axx}(t)} \end{bmatrix} \qquad (7.2)$$

Then the coupled set of vector matrix differential equations for propagating the mean states, $x_{am}(t)$, and the covariances $P_a(t)$ derived in Chapter V can be used to form the nonlinear differential equation

$$\dot{x}(t) = f(t, x(t), w_2(t), \phi) \qquad (7.3)$$

where $x(t)$ and ϕ are as defined above and $w_2(t)$ is the deterministic component of the input disturbance defined in Eq (5.4.1). In the context of Eq (7.2), $w_2(t)$ is regarded as the input or control signal. Note that the control output of the human model, $u(t)$, is one of the components of the state vector $x_a(t)$ and thereby its average value and standard deviation are components of $x(t)$ in Eq (7.3). The observations are viewed as a function of the states $x(t)$; i.e.,

$$y(t_i) = h(t_i, x(t), \phi) \qquad i = 1, 2, \cdots k \qquad (7.4)$$

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From the simulator operation, one obtains data, $z(t_i)$, which are the measured counterparts to the model values $y(t_i)$. The above discussion is intended to put the optimal control model parameter identification problem into the context of identifiability of a nonlinear system as discussed in Chapters II and III. In the next two sections we will use the results of these chapters to examine the local and global identifiability of the optimal control model parameters.

7.1 Local Identifiability of the Optimal Control Model

Recall from Theorem 3.2 that the parameter vector ϕ is locally identifiable under the observation process if

$$M(\phi) = \sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial \phi} \right]^T \left[\frac{\partial y(t_i)}{\partial \phi} \right] \qquad (7.1.1)$$

is nonsingular.

The matrix $M(\phi)$ will be computed in conjunction with the gradient computations used to estimate the model parameter values. In Section 7.4 we will see that, for each of the cases where computations were made, the determinant of $M(\phi)$ was nonzero. If the determinant of $M(\phi)$ is nonzero, then $M(\phi)$ is nonsingular, thereby indicating that the parameter vector ϕ of the optimal control model is locally identifiable under the observation process used for the computations. Recall also from Theorem 4.8 that if we find a critical point, ϕ^*, of the least squares cost functional $J(\phi)$ and

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$M(\phi^*)$ is nonsingular, then this not only assures local identifiability, but also assures that ϕ^* is a local minimizer of $J(\phi)$. This will be discussed further in a later section when computational results are presented.

7.2 Global Identifiability of the Optimal Control Model

In Chapters II and III, sufficient conditions were developed to establish the global identifiability of the parameter vector ϕ. The complexity of the equations of the optimal control model makes it difficult to use these theorems; however, we will use some of the results of Chapter III to draw certain conclusions concerning the global identifiability of ϕ. In the following sections we will examine the global identifiability of each of the parameters of the optimal control model.

7.2.1 Global Identifiability of τ_n

Consider an observation of the form

$$y(t) = H\, x_{am}(t) \qquad (7.2.1)$$

where $x_{am}(t)$ is the vector of mean states given by Eq (5.4.22), i.e.,

$$x_{am}(t) = \hat{x}_{am}(t \mid \sigma) + e^{A_a(\sigma)\tau}\, e_{1m}(\sigma) + e_{2m}(t) \qquad (7.2.2)$$

We will apply Theorem 3.9 to examine the global identifiability of τ_n. Theorem 3.9 uses the recursion relation of Eq (2.1.14) to derive the functions F_0, F_1, $\cdots$, $F_{n+\ell+s}$

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and then the vector $\bar{F}(t)$ is formed as follows:

$$\bar{F}(t) = \begin{bmatrix} F_o^T & F_1^T & \cdots & F^T_{n+\ell+s-1} \end{bmatrix} \qquad (7.2.3)$$

By Theorem 3.9, if Φ is open and path-connected, and if we can find a subvector $\bar{F}_1(t)$ of $\bar{F}(t)$, formed from s components of $\bar{F}(t)$, which is continuously differentiable and such that $\partial\bar{F}_1(t^*)/\partial\phi$ is nonsingular for some $t^* \in [t_o, t_f]$ and all $\phi \in \Phi$, then ϕ is globally identifiable on Φ. In this case $s = 1$ for $\phi = \tau_n$ and we will consider $\Phi = (0, \infty)$ so that Φ is open and path-connected. From the recursion relation Eq (2.1.14)

$$F_o = H\, x_{am}(t)$$

$$F_1 = H\, \dot{x}_{am}(t)$$

$$F_2 = H\, \ddot{x}_{am}(t) \qquad (7.2.4)$$

$$\vdots$$

$$F_{n+\ell+s} = H\, \frac{\partial^{n+\ell+s} x_{am}(t)}{\partial t^{n+\ell+s}}$$

Consider $F_1 = H\, \dot{x}_{am}(t)$. From Eq (7.2.2) we obtain

$$F_1 = H\left[\dot{\hat{x}}_{am}(t \mid \sigma) + e^{A_a(\sigma)\tau}\, \dot{e}_{1m}(\sigma) + \dot{e}_{2m}(t) \right] \qquad (7.2.5)$$

From Eqs (5.4.25) and (5.4.26) we see that $e_{1m}(\sigma)$ and $e_{2m}(t)$ are independent of τ_n. Therefore

$$\frac{\partial F_1}{\partial \tau_n} = H\, \frac{\partial\, \dot{\hat{x}}_{am}(t \mid \sigma)}{\partial\, \tau_n} \qquad (7.2.6)$$

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Using Eq (5.4.24) we obtain

$$\frac{\partial F_1}{\partial \tau_n} = H \frac{\partial}{\partial \tau_n} \left[A_c(t) \hat{x}_{am}(t \mid \sigma) + K(\sigma) C_a(\sigma) e_{lm}(\sigma) \right]$$

$$= H \frac{\partial A_c(t)}{\partial \tau_n} \hat{x}_{am}(t \mid \sigma) \qquad (7.2.7)$$

Recall from Chapter V that the closed loop control matrix $A_c(t)$ is defined by

$$A_c(t) = A_a(t) - B_a \lambda_c$$

$$= \begin{bmatrix} A & \vdots & B(t) \\ -- & \vdots & --- \\ -\lambda' & \vdots & -1/\tau_n \end{bmatrix} \qquad (7.2.8)$$

where

$$\lambda' = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{n+\ell} \end{bmatrix}$$

Thus

$$\frac{\partial A_c(t)}{\partial \tau_n} = \begin{bmatrix} 0 & \vdots & 0 \\ -\frac{\partial \lambda'}{\partial \tau_n} & \vdots & \frac{1}{\tau_n^2} \end{bmatrix} \qquad (7.2.9)$$

This results in

$$\frac{\partial F_1}{\partial \tau_n} = H \begin{bmatrix} 0 & \vdots & 0 \\ -\frac{\partial \lambda'}{\partial \tau_n} & \vdots & \frac{1}{\tau_n^2} \end{bmatrix} \hat{x}_{am}(t) \qquad (7.2.10)$$

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From Eq (7.2.10), it is apparent that if

$$H = \begin{bmatrix} 0 & \cdots & 0 & \vdots & 1 \end{bmatrix} \qquad (7.2.11)$$

then $\dfrac{\partial F_1}{\partial \tau_n}$ is nonsingular (nonzero) and continuously differ-
entiable on the open and path-connected set $(0, \infty)$. There-
fore we can state that τ_n is identifiable on $(0, \infty)$ under
this observation. Also recall that

$$x_a(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad (7.2.11)$$

so that

$$y(t) = \begin{bmatrix} 0 & \vdots & 1 \end{bmatrix} x_{am}(t) \qquad (7.2.12)$$

corresponds to observing the mean control $U_{am}(t)$.

Now consider the simulator dynamics described in Chapter VI.
The sight position, θ_{ST}, is related to the control U_{RA} by
the transfer function

$$\theta_{SA} = \frac{K_2 \, U_{RA}}{s(s^2 + \beta s + \alpha)} \qquad (7.2.13)$$

where $\theta_{ST}(t) = \theta_{SA}(t) \cos \theta_E(t)$ with $0 < \theta_E < \pi/2$ and $s > 0$.
From the transfer function, it is apparent that there is a
one-to-one correspondence between $\theta_{ST}[t_o, t]$ and $U_{RA}[t_o, t]$.
Likewise there is a one-to-one correspondence between the
functions $E\{\theta_{ST}(t)\}$ and $E\{U_{RA}(t)\}$. Since τ_n is identifiable
by observing $E\{U_{RA}(t)\} = U_{RAM}(t)$, then the one-to-one corre-
spondence implies τ_n is identifiable on $(0, \infty)$ by observing

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$\dot{E}\{\Theta_{ST}(t)\} = \Theta_{STM}(t)$. Also we know that the target position $\Theta_T(t)$ is a deterministic quantity defined by the predetermined trajectory. Therefore there is a one-to-one correspondence between the functions $\Theta_{ST}(t)$ and $\Theta_{ST}(t) - \Theta_T(t)$ and also between their mean values. This allows us to state that τ_n is identifiable on $(0, \infty)$ by observing $\dot{E}\{\Theta_{ST} - \Theta_T\}$.

We will now show that we can come to the same conclusion concerning identifiability of τ_n under the observations $\dot{E}\{\Theta_{ST}\}$ and $\dot{E}\{\Theta_{ST} - \Theta_T\}$ by continuing to examine the components F_2, F_3, and F_4 of the recursion relation. Using the same approach for $F_2 = H \ddot{x}_{am}(t)$ as we used for F_1, we obtain

$$\frac{\partial F_2}{\partial \tau_n} = H \frac{\partial \ddot{x}_{am}(t)}{\partial \tau_n}$$

$$= H \frac{\partial}{\partial \tau_n} \left[\frac{\partial^2 \hat{x}_{am}(t|\sigma)}{\partial t^2} \right]$$

$$= H \frac{\partial}{\partial \tau_n} \left[\frac{\partial}{\partial t} \left[A_c(t) \hat{x}_{am}(t|\sigma) \right] \right]$$

$$= H \frac{\partial}{\partial \tau_n} \left[\dot{A}_c(t) \hat{x}_{am}(t|\sigma) + A_c^2(t) \hat{x}_{am}(t|\sigma) \right] \qquad (7.2.14)$$

Using Eq (7.2.8) we can determine

$$A_c^2(t) = \left[\begin{array}{c|c} A - B(t)\lambda' & A B(t) - \frac{1}{\tau_n} B(t) \\ \hline -\lambda'A + \lambda/\tau_n & -\lambda B(t) + 1/\tau_n^2 \end{array} \right] \qquad (7.2.15)$$

so that

$$\frac{\partial A_2^2(t)}{\partial \tau_n} = \begin{bmatrix} -B(t)\frac{\partial \lambda'}{\partial \tau_n} & \vdots & (1/\tau_n^2)B(t) \\ \hline \frac{\partial \lambda'}{\partial \tau_n}A - \frac{\lambda'}{\tau_n^2} + \frac{1}{\tau_n}\frac{\partial \lambda'}{\partial \tau_n} & \vdots & -\frac{\partial \lambda'}{\partial \tau_n}B(t) - 2/\tau_n^3 \end{bmatrix}$$

$$(7.2.16)$$

From Eqs (7.2.14) and (7.2.16) we see that if we observe a mean state $x_{am}(t)$ corresponding to a nonzero component of $B(t)$, then Theorem 3.9 is satisfied and τ_n is observable on $(0, \infty)$. For the simulator described in Chapter VI we have

$$A_c(t) = \begin{bmatrix} A & \vdots & B(t) \\ \hline -\lambda' & \vdots & -1/\tau_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 \\ 0 & -\alpha & -\beta & 0 & \vdots & K_2\cos\theta_E(t) \\ -1 & 1 & 0 & 0 & \vdots & 0 \\ \hline -\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 & \vdots & -1/\tau_n \end{bmatrix} \qquad (7.2.17)$$

and

$$x_{am}(t) = E\left\{ \begin{array}{c} \dot{\Theta}_T \\ \dot{\Theta}_{ST} \\ \ddot{\Theta}_{ST} \\ \Theta_{ST} - \Theta_T \\ U_{RA} \end{array} \right\} \qquad (7.2.18)$$

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Therefore on observation of the form

$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} x_{am}(t) \qquad (7.2.19)$$

will result in τ_n being observable on $(0, \infty)$. This corresponds to observing the mean sight acceleration, $\dot{E}\{\theta_{ST}\}$.

In a similar fashion, we find for $F_3 = H \dddot{x}_{am}(t)$ that

$$\frac{\partial F_3}{\partial \tau_n} = H \frac{\partial}{\partial \tau_n}\left[\ddot{A}_c(t)\hat{x}_{am}(t|\sigma) + 2\dot{A}_c A_c \hat{x}_{am}(t|\sigma) + A_c \dot{A}_c \hat{x}_{am}(t|\sigma) \right.$$

$$\left. + A_c^3(t)\hat{x}_{am}(t|\sigma) \right] \qquad (7.2.20)$$

Also

$$A_c^3(t) = \left[\begin{array}{c|c} A & B(t) \\ \hline -\lambda' & -1/\tau_n \end{array} \right] \left[\begin{array}{c|c} A^2 - B(t)\lambda' & A\,B(t) - 1/\tau_n B(t) \\ \hline -\lambda'A(t) + \lambda'^2 & -\lambda'B(t) + 1/\tau_n^2 \end{array} \right]$$

$$= \left[\begin{array}{c|c} A^3 - A\,B(t)\lambda' - B(t)\lambda'A + B(t)\lambda'^2 & \\ \hline -\lambda'A^2 + \lambda'B(t)\lambda' + \dfrac{\lambda'A}{\tau_n} - \dfrac{\lambda'^2}{\tau_n} & \end{array} \right.$$

$$\left. \begin{array}{c} A^2\,B(t) - \dfrac{A\,B(t)}{\tau_n} - \lambda'B^2(t) + \dfrac{B(t)}{\tau_n^2} \\ \hline -\lambda'A\,B(t) + \dfrac{\lambda'B(t)}{\tau_n} + \dfrac{\lambda'B(t)}{\tau_n} - \dfrac{1}{\tau_n^2} \end{array} \right]$$

$$= \left[\begin{array}{c|c} A_{c1}^3 & A_{c2}^3 \\ \hline A_{c3}^3 & A_{c4}^3 \end{array} \right] \qquad (7.2.21)$$

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so that

$$\frac{\partial}{\partial \tau_n} A_c^3(t) = \begin{bmatrix} -A\, B(t) - B(t)\, \frac{\partial \lambda'}{\partial \tau_n}\, A & \frac{A\, B(t)}{\tau_n^2} - \frac{\partial \lambda'}{\partial \tau_n}\, B^2(t) - \frac{2B(t)}{\tau_n^3} \\ \quad + 2\, B(t)\lambda'\, \partial\lambda'/\partial\tau_n & \\ \hline \frac{\partial A_{c3}^3}{\partial \tau_n} & \frac{\partial A_{c4}^3}{\partial \tau_n} \end{bmatrix}$$

$$(7.2.22)$$

For the simulator of Chapter VI

$$A\, B(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha & -\beta & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ K_2\, \cos\theta_E(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ K_2\, \cos\theta_E(t) \\ -\beta\, K_2\, \cos\theta_E(t) \\ 0 \end{bmatrix} \qquad\qquad (7.2.23)$$

From Eqs (7.2.20), (7.2.22), and (7.2.23) we see that $\partial F_3/\partial \tau_n$ is nonzero and continuously differentiable on $(0, \infty)$ if $y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} x_{am}(t)$. This corresponds to observing the mean sight velocity; $E\{\dot{\theta}_{ST}\}$. Therefore τ_n is identifiable on $(0, \infty)$ under this observation.

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Finally for $F_4 = H \dddot{x}_{am}(t)$, we obtain

$$\frac{\partial F_4}{\partial \tau_n} = H \frac{\partial}{\partial \tau_n} \Big\{ \Big[\dddot{A}_c(t) + 2\ddot{A}_c(t)A_c(t) + 3\dot{A}_c^2(t) + A_c(t)\ddot{A}_c(t)$$
$$+ \dot{A}_c(t)A_c^2(t) + A_c(t)\dot{A}_c(t)A_c(t) + A_c^2(t)\dot{A}_c(t)$$
$$+ A_c^4(t) \Big] \hat{x}_{am}(t|\sigma) \Big\}$$

$$(7.2.24)$$

Also

$$A_c^4(t) = \begin{bmatrix} A^4 - A^2 B(t)\lambda' - A B(t)\lambda'A + A B(t)\lambda'^2 & \vdots \\ -B(t)\lambda'A^2 + B(t)\lambda'B(t)A + \dfrac{B\lambda'A}{\tau_n} - \dfrac{B\lambda'^2}{\tau_n} & \vdots \\ \text{------------------------} & \vdots \\ A_{c3}^4(t) & \vdots \end{bmatrix}$$

$$\begin{bmatrix} A^3 B(t) - \dfrac{A^2 B(t)}{\tau_n} - A\lambda'B^2(t) \\ + \dfrac{A B(t)}{\tau_n^2} - B(t)\lambda'B(t) + \dfrac{B(t)}{\tau_n^2} \\ \text{------------------------} \\ A_{c4}^4(t) \end{bmatrix}$$

$$(7.2.25)$$

so that

$$\frac{\partial A_c^4(t)}{\partial \tau_n} = \begin{bmatrix} \partial A_{c1}^4(t)/\partial\tau_n & \vdots & A^2 B(t)/\tau_n^2 + \text{terms} \\ \text{-----} & \vdots & \text{--------} \\ \partial A_{c3}^4(t)/\partial\tau_n & \vdots & \partial A_{c4}^4(t)/\partial\tau_n \end{bmatrix} \quad (7.2.26)$$

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For the simulator of Chapter VI

$$A^2B(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha & -\beta & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ K_2 \cos\theta_E(t) \\ -\beta \, K_2 \cos\theta_E(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\beta \, K_2 \cos\theta_E(t) \\ \beta \, K_2 \cos\theta_E(t) \\ K_2 \cos\theta_E(t) \end{bmatrix} \qquad\qquad (7.2.27)$$

Eqs (7.2.24), (7.2.26), and (7.2.27) indicate that an observation of the form $y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} x_{am}(t)$ will result in τ_n being identifiable on $(0, \infty)$.

In summary, we have shown that τ_n is globally identifiable on $(0, \infty)$ if the mean control $U_{Rm}(t)$ is observed alone. Also for the simulator of Chapter VI, any of the following observations alone will also result in τ_n being identifiable on $(0, \infty)$.

$$y(t) = H_i \, x_{am}(t)$$

with

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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and

$$x_{am}(t) = E \left\{ \begin{array}{c} \dot{\Theta}_T \\ \dot{\Theta}_{ST} \\ \ddot{\Theta}_{ST} \\ \Theta_{ST} - \Theta_T \\ U_{RA} \end{array} \right\}$$

7.2.2 Global Identifiability of τ

Again consider observations of the form

$$y(t) = H \, x_{am}(t) \qquad (7.2.28)$$

where

$$x_{am}(t) = \hat{x}_{am}(t|\sigma) + e^{A_a(\sigma)\tau} e_{1m}(\sigma) + e_{2m}(t) \qquad (7.2.29)$$

In this case we will use Theorem 3.6 to examine the global identifiability of τ. That is, if Φ is open and path-connected and there exists a component of $y(t)$ denoted by $y_1(t)$ which is continuously differentiable and $\partial y_1(t^*)/\partial \tau$ is nonsingular (nonzero) for $t^* \in [t_o, t_f]$, and all $\phi \in \Phi$, then τ is identifiable on Φ.

From Eq (7.2.29),

$$\frac{\partial y(t)}{\partial \tau} = H \left[\partial \hat{x}_{am}(t|\sigma)/\partial \tau + A_a(\sigma) e^{A_a(\sigma)\tau} e_{1m}(\sigma) + \partial e_{2m}(t)/\partial \tau \right]$$

$$(7.2.30)$$

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Consider the second term of Eq (7.2.30). From Chapter VI we know the simulator dynamics are described by

$$A(\sigma) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\alpha & -\beta & 0 & K_2 \cos\theta_E(\sigma) \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\tau_n \end{bmatrix} \qquad (7.2.31)$$

Also

$$e^{A_a(\sigma)\tau} = I + A_a\tau + \frac{A_a^2\tau^2}{2!} + \cdots \qquad (7.2.32)$$

One can inductively show that

$$A_a^n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A_{32}^{n-1} & A_{33}^{n-1} & 0 & A_{43}^{n-1} \\ 0 & A_{32}^1 A_{33}^{n-1} & A_{32}^{n-1}-A_{32}^1 A_{33}^{n-1} & 0 & A_{35}^1 A_{33}^{n-1}+A_{55}^1 A_{35}^{n-1} \\ 0 & A_{22}^{n-1} & A_{23}^{n-1} & 0 & A_{25}^{n-1} \\ 0 & 0 & 0 & 0 & A_{55}^1(A_{55}^{n-1}) \end{bmatrix}$$

$$(7.2.33)$$

for $n \geq 2$

Thus from Eqs (7.2.32) and (7.2.33) one can state that the following elements of $e^{A_a(\sigma)\tau}$ will be nonzero:

$$e^{A_a(\sigma)\tau} = \begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 \\ 0 & E_{21} & E_{23} & 0 & E_{25} \\ 0 & E_{32} & E_{33} & 0 & E_{35} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} \\ 0 & 0 & 0 & 0 & E_{55} \end{bmatrix} \qquad (7.2.34)$$

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Then

$$A_a(\sigma) e^{A_a(\sigma)\tau} e_{1m}(\sigma)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\alpha & -\beta & 0 & K_2 \cos\Theta_E(\sigma) \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\tau_n \end{bmatrix} e^{A_a(\sigma)\tau} e_{1m}(\sigma)$$

$$(7.2.35)$$

with $e^{A_a(\sigma)\tau}$ having the form of Eq (7.2.34). From Eq (7.2.35), we see that observations of the form

$$y(t) = H\, x_{am}(t)$$

with

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$
$$H = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

will result in τ being identifiable on $(0, \infty)$. We cannot use Eq (7.2.35) to state identifiability under the observation $H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ because the mean estimator error of the control (i.e., the fifth component of $e_{1m}(\sigma)$)

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is zero. However we can show identifiability under this observation by referring to the transfer function for the simulator.

Recall that the sight position, Θ_{ST}, is related to the control U_{RA} by

$$\Theta_{SA} = \frac{K_2 \, U_{RA}}{s(s^2 + \beta s + \alpha)} \qquad (7.2.36)$$

where $\Theta_{ST}(t) = \Theta_{SA}(t) \, \cos\Theta_E(t)$ with $0 < \Theta_E < \pi/2$ and $s > 0$. Thus there is a one-to-one correspondence between the functions $\Theta_{ST}(t)$ and $U_{RA}(t)$ and also between the ensemble averages of these functions; i.e., $E\{\Theta_{ST}(t)\}$ and $E\{U_{RA}(t)\}$. Since $\Theta_T(t)$ is a deterministic function independent of τ, this allows us to state that there is a one-to-one correspondence between the functions $E\{\Theta_{ST} - \Theta_T\}$ and $\dot{E}\{U_{RA}(t)\}$.

—

Therefore, since τ is identifiable on $(0, \infty)$ by observing $\dot{E}\{\Theta_{ST}(t) - \Theta_{T}(t)\}$, we can state that τ is identifiable on $(0, \infty)$ by observing $\dot{E}\{U_{RA}(t)\}$.

In summary, τ is identifiable on $(0, \infty)$ for the simulator system described in Chapter VI under any of the following observations

$$y(t) = H_i \, x_{am}(t)$$

where

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

7.2.3 Global Identification of ρ_i

Let $P_{axx}(t)$ be the variances of the states $x_a(t)$. Now consider observations of the form

$$y(t) = \sqrt{H \, P_{axx}(t)} \qquad (7.2.37)$$

To examine the global identifiability of ρ_i, the proportionality constants for calculating observation noise variance kernels, we will again use Theorem 3.9. Let Φ be the open and path-connected set $(0, \infty)$ and again let

$$\bar{F}(t) = \begin{bmatrix} F_o^T & F_1^T & \cdots & F_{n+\ell+s-1} \end{bmatrix}^T \qquad (7.2.38)$$

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where

$$F_0 = y(t)$$

$$F_1 = \dot{y}(t)$$

$$F_2 = \ddot{y}(t)$$

$$\vdots$$

$$F_{n+\ell+s} = \frac{\partial^{n+\ell+s} y(t)}{\partial t^{n+\ell+s}}$$

ρ_i is identifiable on ϕ if we can find a component of $\bar{F}(t)$, say $\bar{F}_1(t)$, which is continuously differentiable and $\partial \bar{F}_1(t^*)/\partial \rho_i$ is nonzero for some $t^* \in \left[t_o, t_f \right]$ and all $\rho_i \in (0, \infty)$. From Eq (5.4.32), we know

$$P_a(t) = e^{A_a(\sigma)\tau} P_2(\sigma) e^{A_a^T(\sigma)\tau} + P_3 e^{A_a^T(\sigma)\tau}$$

$$+ e^{A_a(\sigma)\tau} P_3^T(t) + P_4(t) + P_5(t) \qquad (7.2.39)$$

Using Eqs (5.4.28) to (5.4.31) we obtain the following as a first step to evaluating $\partial \bar{F}_1(t)/\partial \rho_i$.

$$\frac{\partial \dot{P}_a(t)}{\partial \rho_i} = \frac{\partial}{\partial \rho_i} \left[e^{A_a(\sigma)\tau} \dot{P}_2 e^{A_a^T(\sigma)\tau} + \dot{P}_3 e^{A_a^T(\sigma)\tau} \right.$$

$$\left. + e^{A_a(\sigma)\tau} \dot{P}_3^T(t) + \dot{P}_4(t) \right] \qquad (7.2.40)$$

$$\frac{\partial \dot{P}_2}{\partial \rho_i} = \frac{\partial A_f}{\partial \rho_i} P_2 + P_2 \frac{\partial A_f^T}{\partial \rho_i} + P_1 C_a^T \frac{\partial V^{-1}}{\partial \rho_i} C_a P_1 \qquad (7.2.41)$$

$$\frac{\partial A_f}{\partial \rho_i} = - P_1 C_a^T \frac{\partial V^{-1}}{\partial \rho_i} C_a \qquad (7.2.42)$$

$$\frac{\partial \dot{P}_3}{\partial \rho_i} = P_3 \frac{\partial A_f^T}{\partial \rho_i} + \frac{\partial K}{\partial \rho_i} C_a \left[P_2 - P_1 \right] \tag{7.2.43}$$

$$\frac{\partial K}{\partial \rho_i} = e^{A_a \tau} \frac{\partial G}{\partial \rho_i} = e^{A_a \tau} P_1 C_a^T \frac{\partial V^{-1}}{\partial \rho_i} \tag{7.2.44}$$

$$\frac{\partial \dot{P}_4}{\partial \rho_i} = \frac{\partial K}{\partial \rho_i} C_a^T P_3^T + P_3 C_a^T \frac{\partial K^T}{\partial \rho_i} + e^{A_a \tau} P_1 C_a^T \frac{\partial V^{-1}}{\partial \rho_i} C_a P_1 e^{A_a^T \tau} \tag{7.2.45}$$

Now consider the simulator of Chapter VI where

$$C_a = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$V^{-1} = \begin{bmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{bmatrix}$$

Thus

$$C_a^T V^{-1} C_a = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} V_2^{-1} & -V_2^{-1} & 0 & 0 & 0 \\ -V_2^{-1} & V_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_1^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Now $P_1 C_a{}^T V^{-1} C_a P_1{}^T$, which is a term of $\dot{P}_2$, will have the form $P_1 C_a{}^T V^{-1} C_a P_1 =$

$$
\begin{bmatrix}
a_{11} V_2^{-1} & a_{12} V_2^{-1} & a_{13} V_2^{-1} & a_{14}^1 V_1^{-1} + a_{14}^2 V_2^{-1} & a_{15} V_2^{-1} \\
a_{21} V_2^{-1} & a_{22} V_2^{-1} & a_{23} V_2^{-1} & a_{24}^1 V_1^{-1} + a_{24}^2 V_2^{-1} & a_{25} V_2^{-1} \\
a_{31} V_2^{-1} & a_{32} V_2^{-1} & 0 & 0 & 0 \\
a_{41}^1 V_1^{-1} + a_{41} V_2^{-1} & a_{42}^1 V_1^{-1} + a_{42} V_2^{-1} & 0 & a_{44} V_1^{-1} & 0 \\
a_{51} V_2^{-1} & a_{52} V_2^{-1} & 0 & 0 & 0
\end{bmatrix}
$$

From Eq (7.2.34) we know

$$
e^{A_a \tau} =
\begin{bmatrix}
E_{11} & 0 & 0 & 0 & 0 \\
0 & E_{21} & E_{23} & 0 & E_{25} \\
0 & E_{32} & E_{33} & 0 & E_{35} \\
E_{41} & E_{42} & E_{43} & E_{44} & E_{45} \\
0 & 0 & 0 & 0 & E_{55}
\end{bmatrix}
$$

Now consider $J \triangleq e^{A_a \tau} P_1 C_a{}^T V^{-1} C_a P_1{}^T e^{A_a{}^T \tau}$. From the above, it is seen that the diagonal element J_{44} will have the form

$$
J_{44} = e_{44}^1 V_1^{-1} + e_{44}^2 V_2^{-1}
$$

with e_{44}^1 and e_{44}^2 nonzero.

Also recall that

$$V(t) = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} C_a P_a(t) C_a^T$$

so that

$$\frac{\partial V^{-1}(t)}{\partial \rho_i} = - \frac{1}{\rho_i} V^{-1}(t)$$

Combining the above results with Eqs (7.2.40) through (7.2.45) we see that $\partial \dot{P}_a(t)/\partial \rho_i$ will have a nonzero diagonal term $\partial \dot{P}_{a44}(t)/\partial \rho_i$. This means observations of the form

$$y(t) = \sqrt{H \, P_{axx}(t)} \qquad \text{with } H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

will result in ρ_1 and ρ_2 being identifiable on $(0, \infty)$. This form of observation satisfies Theorem 3.9 since

$$\frac{\partial F_1}{\partial \rho_i} = \frac{\partial}{\partial \rho_i} \dot{y}(t) = \frac{1}{2} \frac{\partial}{\partial \rho_i} \left[\frac{1}{P_{a44}(t)} \quad \dot{P}_{a44}(t) \right]$$

$$= \frac{1}{2} \frac{1}{P_{a44}(t)} \frac{\partial \dot{P}_{a44}(t)}{\partial \rho_i} - \frac{1}{2} \frac{1}{P_{a44}^2(t)} \dot{P}_{a44}(t) \frac{\partial P_{a44}(t)}{\partial \rho_i}$$

will be nonzero for $\rho_i \, \epsilon \, (0, \infty)$, $i = 1, 2$.

Therefore, for the simulator dynamics described in Chapter VI, the parameters ρ_i ($i = 1, 2$) are identifiable

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on the open interval $(0, \infty)$ under an observation

$$y(t) = \sqrt{H \, P_{axx}(t)} \qquad \text{with } H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This corresponds to observing the standard deviation of the tracking error, $\Theta_{ST} - \Theta_T$.

Now recall from our previous discussion that

$$\Theta_{SA} = \frac{K_2 \, U_{RA}}{s(s^2 + \beta s + \alpha)}$$

where $\Theta_{ST}(t) = \Theta_{SA}(t) \cos\Theta_E(t)$ with $0 < \Theta_E < \pi/2$. This implies there is a one-to-one correspondence between the functions $\Theta_{ST}(t)$ and $U_{RA}(t)$; and thus between the functions $\Theta_{ST}(t)$ and $U_{RA}(t)$. Since ρ_i is identifiable on $(0, \infty)$ by observing the standard deviation of $\Theta_{ST}(t) - \Theta_T(t)$ and since $\Theta_T(t)$ is a deterministic function independent of ρ_i, we can state that ρ_i is identifiable on $(0, \infty)$ by observing the standard deviation of $U_{RA}(t)$.

7.2.4 Global Identification of W_1 and ρ_m

Again consider observations of the form

$$y(t) = \sqrt{H \, P_{axx}(t)}$$

Using the same approach as for ρ_i, we see that finding nonzero components of $\partial \dot{P}_a(t)/\partial \phi_i$ ($\phi_1 = W_1$ and $\phi_2 = \rho_m$) depends on examining the term

$$e^{A_a \tau} \, \Gamma_a \begin{bmatrix} W_1 & 0 \\ 0 & V_m \end{bmatrix} \Gamma_a^T \, e^{A_a^T \tau}.$$

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Again consider the simulator of Chapter VI, where

$$\Gamma_a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1/\tau_n \end{bmatrix}$$

so that

$$\Gamma_a \begin{bmatrix} W_1 & 0 \\ & \\ 0 & V_m \end{bmatrix} \Gamma_a^{\ T} = \begin{bmatrix} W_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_m/\tau_n^{\ 2} \end{bmatrix}$$

Also from Eq (7.2.34)

$$e^{A_a\tau} = \begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 \\ 0 & E_{21} & E_{23} & 0 & E_{25} \\ 0 & E_{32} & E_{33} & 0 & E_{35} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} \\ 0 & 0 & 0 & 0 & E_{55} \end{bmatrix}$$

so we obtain

$$e^{A_a\tau}\Gamma_a \begin{bmatrix} W_1 & 0 \\ & \\ 0 & V_m \end{bmatrix} \Gamma_a^{\ \tau} e^{A_a^T\tau} =$$

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\ J_{41} & J_{42} & J_{43} & J_{44}W_1 & J_{45} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55}V_m \end{bmatrix} \tag{7.2.46}$$

where J_{44} and J_{55} are nonzero scalars.

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This implies that $\partial \dot{P}_{a44}(t)/\partial W_1$ is nonzero. Also $\partial \dot{P}_{55}(t)/\partial \rho_m$ is nonzero. Using Theorem 3.9 and the same approach as for ρ_i we conclude that W_1 is identifiable on $(0, \infty)$ under the observation

$$y(t) = \sqrt{H\ P_{axx}(t)} \qquad \text{with } H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Again this corresponds to observing the standard deviation of the tracking error.

Also ρ_m is identifiable on $(0, \infty)$ under the observation

$$y(t) = \sqrt{H\ P_{axx}(t)}$$

with $H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

This corresponds to observing the standard deviation of the sight acceleration, $\ddot{\Theta}_{ST}$, or the standard deviation of the control, U_{RA}.

Again recall that the sight position Θ_{ST} is related to the control U_{RA} by the transfer function

$$\Theta_{SA} = \frac{K_2\ U_{RA}}{s(s^2 + \beta s + \alpha)}$$

where $\Theta_{ST}(t) = \Theta_{SA}(t)\cos\Theta_E(t)$ with $0 < \Theta_E < \pi/2$. This implies a one-to-one correspondence between the functions $\Theta_{ST}(t)$ and $U_{RA}(t)$. Also this assures a one-to-one correspondence between the functions $\Theta_{ST}^2(t)$ and $U_{RA}^2(t)$ and thus between the standard deviation of Θ_{ST} and U_{RA}. Now since ρ_m is identifiable under the observation of the standard

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deviation of $U_{RA}(t)$, we also can state that ρ_m is identifiable under the observation of the standard deviation of $\Theta_{ST}(t)$. Since the target position is a deterministic quantity, the standard deviation of the tracking error, $\Theta_{ST}(t)-\Theta_T(t)$, is equal to the standard deviation of the sight position. Thus ρ_m is also identifiable under the observation of the standard deviation of the tracking error.

Using a similar argument, we can state that W_1 is identifiable on $(0, \infty)$ by observing the standard deviation of the function $U_{RA}(t)$.

In summary, both $\rho_i (i = 1, 2)$ and W_1 are identifiable on $(0, \infty)$ by observing either the standard deviation of $U_{RA}(t)$ or $\Theta_{ST}(t) - \Theta_T(t)$.

7.2.5 Global Identification of τ_c

Recall that in Section 5.4 an alternative model for the plant disturbance was presented. The alternative has the white noise component equal to zero and thus the plant disturbance is

$$w(t) = w_2(t) \tag{7.2.47}$$

and the variance of the disturbance is modeled by

$$W(t) = 2 \, \tau_c \, w_2^2(t) \tag{7.2.48}$$

where τ_c may be viewed as a correlation time for $w_2(t)$. One could assign a value to τ_c equal to an estimate of the

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correlation time or one could attempt to estimate its value using experimental data. In case of the latter, we are interested in this parameter's identifiability.

Following the same approach as in preceding sections we find

$$\frac{\partial \dot{P}_1(t)}{\partial \tau_c} = \Gamma_a \frac{\partial W(t)}{\partial \tau_c} \Gamma_a^T \qquad (7.2.49)$$

where

$$\frac{\partial W(t)}{\partial(t)} = \begin{bmatrix} 2\,w_2^{\,2}(t) & \vdots & 0 \\ - - - - - & \vdots & - - \\ 0 & \vdots & 0 \end{bmatrix}$$

This term couples into $P_2(t)$ by the term

$$A_f(t) = A_a(t) - P_1 C_a^T V^{-1}(\sigma)$$

Using the same approach as for ρ_i and W_1 we can show that τ_c is identifiable on $(0, \infty)$ under an observation of the standard deviation of the tracking error $\Theta_{ST} - \Theta_T$ or of the control, U_{RA}.

7.2.6 Summary of Global Identifiability

In the previous sections we examined the identifiability of the optimal control model parameters under observations of the form

$$y(t) = \begin{bmatrix} H_1 & \vdots & H_2 \end{bmatrix} \begin{bmatrix} x_{am}(t) \\ - - - - \\ \sqrt{P_{axx}(t)} \end{bmatrix}$$

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where

$$x_{am}(t) = \dot{E}\{[\dot{\Theta}_T \quad \dot{\Theta}_{ST} \quad \ddot{\Theta}_{ST} \quad \Theta_{ST} - \Theta_T \quad U_{RA}]^T\}$$

and $\sqrt{P_{axx}(t)}$ is the standard deviation of these states.
For the simulator described in Chapter VI, we show that
the parameters were identifiable on $(0, \infty)$ under the observations indicated by the following matrix:

	Mean States				Standard Deviations	
	$\dot{\Theta}_{ST}$	$\ddot{\Theta}_{ST}$	$\Theta_{ST} - \Theta_T$	U_{RA}	$\Theta_{ST} - \Theta_T$	U_{RA}
τ_n	X	X	X	X		
τ	X	X	X	X		
ρ_1					X	X
ρ_2					X	X
$W_1(\tau_c)$					X	X
ρ_m					X	X

Thus from the theory developed in Chapter III, we know that
$\phi = [\tau_n \quad \tau \quad \rho_1 \quad \rho_2 \quad W_1 \quad \rho_m]^T$ is identifiable under observations of the form of Eq (7.2.50) with

$H = [0 \; 0 \; 0 \; 1 \; 0 \; \vdots \; 0 \; 0 \; 0 \; 1 \; 0]$ or $H = [0 \; 0 \; 0 \; 0 \; 1 \; \vdots \; 0 \; 0 \; 0 \; 0 \; 1]$.

7.3 Computational Results

In the preceding sections we have shown that the parameters of the optimal control model can be identified from
observations of the mean and standard deviation of either
the tracking errors or the manual control inputs to the

tracking system. In this section the results of computations which were performed to identify the parameter values will be presented. We obtain data from actual system operation using the simulator system described in Chapter VI. This yields the data shown in Figs. 6.8.1 through 6.8.4, which results from ensemble averaging many time histories of tracking errors. This simulator-generated data, denoted by $z(t_i)$, $t_i \varepsilon [t_o, t_f]$, has corresponding data resulting from model calculations which are denoted by $y(t_i)$. We want to find an estimate of the parameter vector ϕ which minimizes the cost functional

$$J = \sum_{i=1}^{k} \left[z(t_i) - y(t_i) \right]^T \left[z(t_i) - y(t_i) \right] \qquad (7.3.1)$$

where $\phi = \left[\tau_n \ \tau \ \rho_1 \ \rho_2 \ \rho_m \ W_1 \right]^T$.

The measured data $z(t_i)$ to be used in the identification computations consists of the mean and standard deviations of the tracking errors. As discussed in the previous sections, this is sufficient to allow identification of the parameter vector ϕ. Other sets of measured data could be considered, such as the mean and standard deviation of the control input or combinations of tracking error and control input data. In this research, we will confine our computations to those using tracking error data; follow on research could make use of other possibilities for measured data sets.

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We will use the gradient method for minimizing the cost functional as discussed in Section 4.4.1. This requires calculation of the gradient equations $\partial y(t)/\partial \phi$. Since the measured data is a function of the mean states, $x_{am}(t)$, and their covariances, $P_a(t)$, we need to compute

$$\frac{\partial x_{am}(t)}{\partial \phi} = \left[\frac{\partial x_{am}(t)}{\partial \phi_1} \; \middle| \; \frac{\partial x_{am}(t)}{\partial \phi_2} \; \middle| \; \cdots \; \middle| \; \frac{\partial x_{am}}{\partial \phi_6} \right] \qquad (7.3.2)$$

and

$$\frac{\partial P_a(t)}{\partial \phi} = \left[\frac{\partial P_a(t)}{\partial \phi_1} \; \middle| \; \frac{\partial P_a(t)}{\partial \phi_2} \; \middle| \; \cdots \; \middle| \; \frac{\partial P_a(t)}{\partial \phi_6} \right] \qquad (7.3.3)$$

This is a very tedious operation that follows in a straight-forward manner from the equations for propagating the mean states and covariances. Appendix F summarizes the results of gradient equation evaluation for the simulator under discussion.

Computer programs were developed to propagate the mean states and covariances described by the coupled matrix differential equations in Section 6.5. Also the gradient equations developed in Appendix F are solved using the digital computer. Appendix G presents a flow diagram of the programs and subroutines used to make the required computations.

The gradient method was used to adjust the parameter vector ϕ treating each axis as an independent case. Recall

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that the particular simulator used in this research has two people in the tracking system; one operates the azimuth control and one operates the elevation control. This two-operator system supports the assumption of treating the axes separately (other than the $\cos\theta_E(t)$ term appearing in the traverse equations); however, it is likely that some coupling effects do occur. This will be discussed in conjunction with some of the results to be presented and could be addressed in future research as an important refinement to the model.

In the next sections the results obtained from the parameter identification computations will be presented as well as a comparison of model and simulator data. Recall that two trajectories were considered; the profiles are shown in Figs. 6.7.1 and 6.7.2 respectively. As discussed in Section 6.7, the acquisition phase takes place during the first 10 seconds of the trajectory. This portion of the trajectory was avoided for identification purposes because the assumption of the least squared error cost functional for the trackers would be suspect. To allow any transients remaining from the acquisition phase to damp out, the identification calculations were started at 15 seconds into the trajectories and were computed over 10 seconds to a final time of 25 seconds. This covered the crossover condition ($x = 0$) for both trajectories and was considered adequate to provide a basis for comparing model and

simulator data, as well as for the gradient parameter
identification procedure.

7.3.1 Trajectory 1, Azimuth Axis

An initial value for the vector ϕ was assumed which
corresponds to nominal values used by Kleinman, et al
(Refs 5, 20, 21, 22, 23). These values were

$$\tau_n = .1$$

$$\tau = .2$$

$$\rho_1 = .0314$$

$$\rho_2 = .0314$$

$$\rho_m = .01$$

$$W_1 = .000001$$

Note that the plant disturbance noise, W_1, was introduced
as a new parameter in this research to account for system
noise and model uncertainities, so that no nominal value
was available from the literature. The initial estimate
for W_1 corresponded to an estimate for a nominal value of
$\left[\ddot{\theta}_T(t) \right]^2$ for the trajectory.

Recall from Eq (4.4.10) that the change in ϕ for each
iteration is given by

$$\Delta\phi^{k+1} = \phi^k - \left[N^k \right]^{-1} \left[\frac{\partial J(\phi^k)}{\partial \phi^k} \right]^T \tag{7.3.4}$$

where N^k is a symmetric positive definite matrix to adjust
the step size with the following form

$$N^k = \begin{bmatrix} N_1{}^k & 0 & 0 & 0 & 0 & 0 \\ 0 & N_2{}^k & 0 & 0 & 0 & 0 \\ 0 & 0 & N_3{}^k & 0 & 0 & 0 \\ 0 & 0 & 0 & N_4{}^k & 0 & 0 \\ 0 & 0 & 0 & 0 & N_5{}^k & 0 \\ 0 & 0 & 0 & 0 & 0 & N_6{}^k \end{bmatrix}$$

with

$$J(\phi_k) = \sum_{i=1}^{k} \left[z(t_i) - y(t_i) \right]^T \left[z(t_i) - y(t_i) \right] \qquad (7.3.5)$$

then

$$\frac{\partial J(\phi_k)}{\partial \phi^k} = -2 \left[\sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial \phi^k} \right]^T \left[z(t_i) - y(t_i) \right] \right] \qquad (7.3.6)$$

As stated above $z(t_i)$ is a two element vector

$$z(t\ ) = \begin{bmatrix} z_1(t_i) \\ z_2(t_i) \end{bmatrix} \qquad (7.3.7)$$

where

$z_1(t_i)$ = measured mean tracking error at t_i

$z_2(t_i)$ = standard deviation of measured tracking error at t_i

Table 7.3.1 summarizes the results of using the gradient technique to adjust the values of ϕ for the azimuth axis of trajectory 1.

The following observations are made from Table 7.3.1:

(1) The cost functional $J(\phi^k)$ is reduced at each

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Table 7.3.1

SUMMARY OF GRADIENT ITERATIONS
TRAJECTORY 1, AZIMUTH AXIS

k=1	ϕ^k	$-\frac{1}{2}\frac{\partial J(\phi^k)}{\partial \phi^k}$	$2\left[N^k\right]^{-1}$
$J(\phi^1)=.946\times10^{-4}$	.100	$.1031\times10^{-3}$	50
	.2000	$.5013\times10^{-4}$	50
$\|M(\phi^1)\|=.579\times10^{-16}$	.0314	$.4082\times10^{-3}$	50
	.0314	$.9632\times10^{-4}$	50
	.0100	$.2492\times10^{-3}$	50
	.000001	$.2009\times10^{1}$	5×10^{-6}
k=2			
$J(\phi^2)=.335\times10^{-4}$	.105	$.4236\times10^{-4}$	50
	.2025	$.3307\times10^{-4}$	50
$\|M(\phi^2)\|=.469\times10^{-20}$	.0515	$.6250\times10^{-4}$	50
	.0362	$.1122\times10^{-3}$	50
	.0225	$.1017\times10^{-3}$	50
	.00001104	$.2084\times10^{-2}$	5×10^{-6}
k=3			
$J(\phi^3)=.303\times10^{-4}$	.107	$.1028\times10^{-4}$	10
	.2042	$.1367\times10^{-4}$	10
$\|M(\phi^3)\|=.375\times10^{-20}$	.0546	$.3208\times10^{-4}$	10
	.0418	$.4888\times10^{-4}$	10
	.0283	$.3034\times10^{-4}$	10^{-5}
	.00001105	$-.1019$	10
k=4			
$J(\phi^4)=.288\times10^{-4}$	.108	$.1736\times10^{-5}$	
	.2055	$.7725\times10^{-5}$	
$\|M(\phi^4)\|=.339\times10^{-20}$	.0578	$.2469\times10^{-4}$	
	.0467	$.3088\times10^{-4}$	
	.0313	$.1478\times10^{-4}$	
	.00001003	$-.1404$	

iteration indicating the gradient method for adjusting ϕ was functioning properly.

(2) The values of the diagonal of $\left[N^k \right]^{-1}$ were selected off line to give reasonable step sizes. Since the cost functional was reduced at each step, this indicates the choices for $\left[N^k \right]^{-1}$ were satisfactory. Recall from Section 4.4 that too large a step size can cause overshoot and an increase in the cost functional.

(3) The values of the gradients $\partial J/\partial \phi^k$ decreased with each iteration as one expects. It was found that after four iterations the gradients were sufficiently small to make the change in ϕ^k very slight. The stopping condition was based on $||\Delta \phi^k||$ being less than a preselected small value.

(4) Recall from Eq (7.1.1 that the parameter vector ϕ is locally identifiable if $M(\phi) = \sum\limits_{i=1}^{N} \left[\partial y(t_i)/\partial \phi \right]^T$ $\left[\partial y(t_i)/\partial \phi \right]$ is nonsingular.

The determinant of $M(\phi^k)$, denoted by $|M(\phi^k)|$ was evaluated at each iteration and appears in Table 7.3.1. In each case $|M(\phi^k)|$ is nonzero indicating that $M(\phi)$ is nonsingular. This supports the hypothesis that the parameter vector ϕ is locally identifiable at each of the iteration values, ϕ^k. This suggests the possibility of using a second mode of Newton Raphson iterations to converge rapidly near the end.

Fig. 7.3.1 compares the model data to that obtained from the simulator for the parameters from the fourth iteration

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FIGURE 7.3.1
TRAJECTORY 1, AZIMUTH AXIS
MODEL AND SIMULATOR TRACKING ERRORS

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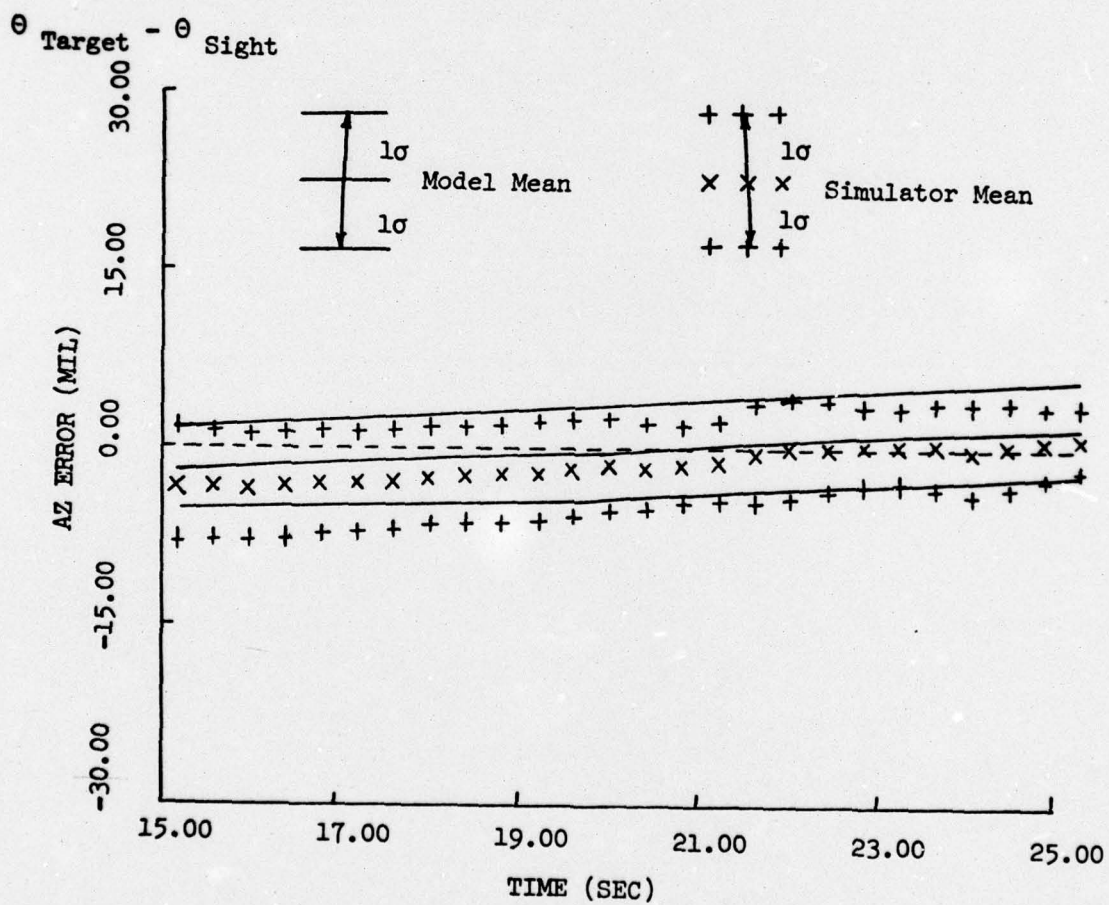
of the azimuth axis identification for trajectory 1. The model mean tracking error is slightly less than the simulator error prior to target crossover at 20 seconds. However, the model results agree with the simulator in that the sight tends to lead the target prior to crossover. This supports the configuration of the model treating target angular acceleration as a plant disturbance as opposed to the third derivative of angular position for example. Under this configuration, the operator (as modeled by the Kalman filter/predictor) estimates the target velocity based on the assumption that acceleration is a zero mean random disturbance. Fig. 7.3.2 shows the actual target angular velocity and acceleration as a function of time.

The value of W_1 (strength of the white noise portion of the plant disturbance) accounts for system noise, model uncertainities, and the ability of the operator to adapt to different trajectories. Increasing W_1 will result in the Kalman filter gain increasing and thereby tend to increase the weight of current observations of the operator. Thus the value of W_1, together with the operator's estimate of the value of the angular acceleration, reflect the adaptability of the model filter and, as such, the operator adaptability. Such techniques have been used in other applications as discussed in Refs 15, 16, and 34.

FIG. 7.3.2
TRAJECTORY 1, AZIMUTH AXIS
ANGULAR VELOCITY AND ACCELERATION
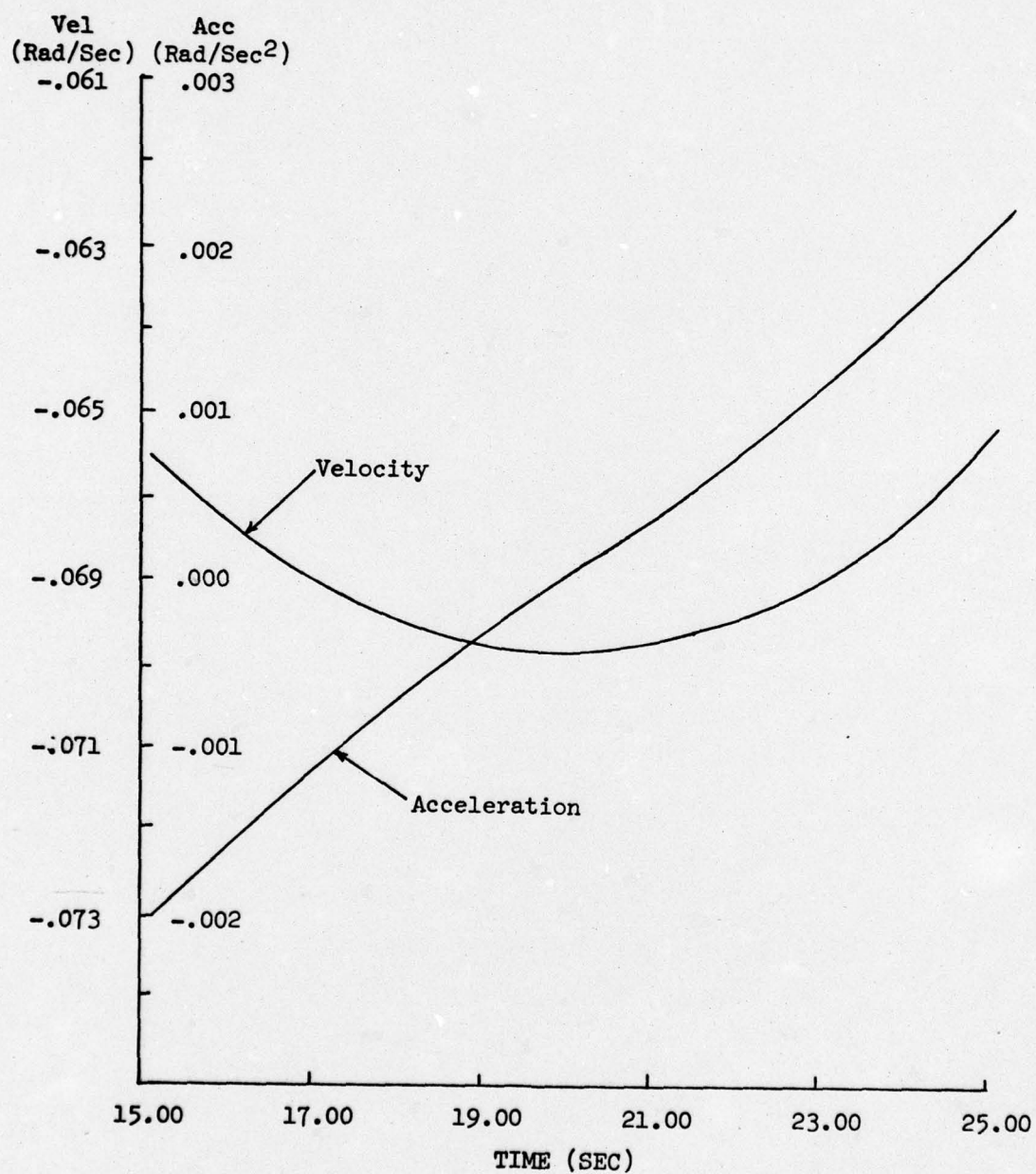
7.3.2 Trajectory 1, Elevation Axis

Table 7.3.2 presents a summary of the gradient calcu-
lations for the elevation axis of trajectory 1. Again the
cost functional is reduced at each iteration. Moreover
$|M(\phi^k)|$ is nonzero indicating local identifiability.
Figure 7.3.3 shows the model and simulator response with
the model data corresponding to the parameter values after
the third iteration of the identification procedure. From
the figure it is apparent that the elevation tracking task
is not very difficult for trajectory 1. The mean response
of the simulator and the model agree in that the sight tends
to be above the target. Again this tends to support the
model configuration with target acceleration as a plant
disturbance. The actual target angular velocity and accel-
eration for the elevation axis of trajectory 1 is shown in
Fig. 7.3.4. The mean tracking error introduced by the tar-
get acceleration in the model response results in general
agreement of model and simulator response.

The simulator mean tracking error has a relatively
small oscillatory characteristic near crossover not present
in the model response. This could be attributed to the
changing image of the target on the tracking monitor during
the crossover phase, with the operator adjusting the ele-
vation cross hair to account for what is perceived as a
small change in elevation. For a deterministic trajectory,
this would be a consistent effect for a given trajectory
and therefore appear as a perturbation to the mean tracking

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Table 7.3.2

SUMMARY OF GRADIENT ITERATIONS
TRAJECTORY 1, ELEVATION AXIS

k=1	ϕ^k	$-\dfrac{1}{2}\dfrac{\partial J(\phi^k)}{\partial \phi^k}$	$2\left[N^k\right]^{-1}$
$J(\phi^1)=.461\times10^{-4}$	.100	$.8140\times10^{-4}$	50
	.2000	$.1064\times10^{-4}$	50
$\|M(\phi^1)\|=.465\times10^{-19}$	.0314	$.1034\times10^{-3}$	50
	.0314	$.4654\times10^{-4}$	50
	.0100	$.2331\times10^{-3}$	50
	.00001	.6021	$.5\times10^{-5}$
k=2			
$J(\phi^2)=.378\times10^{-4}$	.104	$-.5019\times10^{-5}$	10
	.2005	$-.5228\times10^{-4}$	10
$\|M(\phi^2)\|=.292\times10^{-28}$	.0366	$-.5203\times10^{-5}$	10
	.0337	$-.1139\times10^{-3}$	10
	.0216	$-.3911\times10^{-4}$	10
	.00001301	$-.5538$	$.5\times10^{-5}$
k=3			
$J(\phi^3)=.370\times10^{-4}$	.104	$-.5425\times10^{-5}$	
	.2000	$-.5119\times10^{-4}$	
$\|M(\phi^3)\|=.290\times10^{-28}$	.0366	$-.4673\times10^{-5}$	
	.0327	$-.1144\times10^{-3}$	
	.0214	$-.4150\times10^{-4}$	
	.00001352	.5293	

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FIGURE 7.3.3
TRAJECTORY 1, ELEVATION AXIS
MODEL AND SIMULATOR TRACKING ERRORS
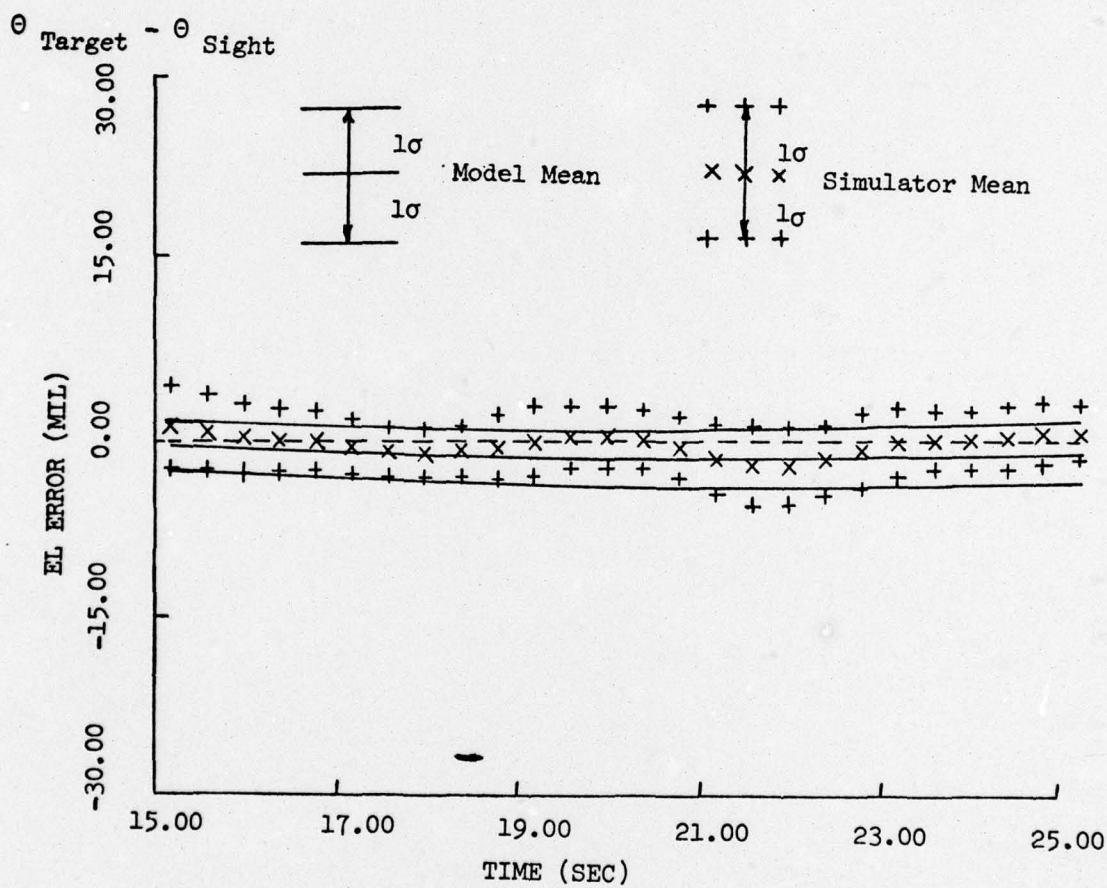
FIG. 7.3.4
TRAJECTORY 1, ELEVATION AXIS
ANGULAR VELOCITY AND ACCELERATION
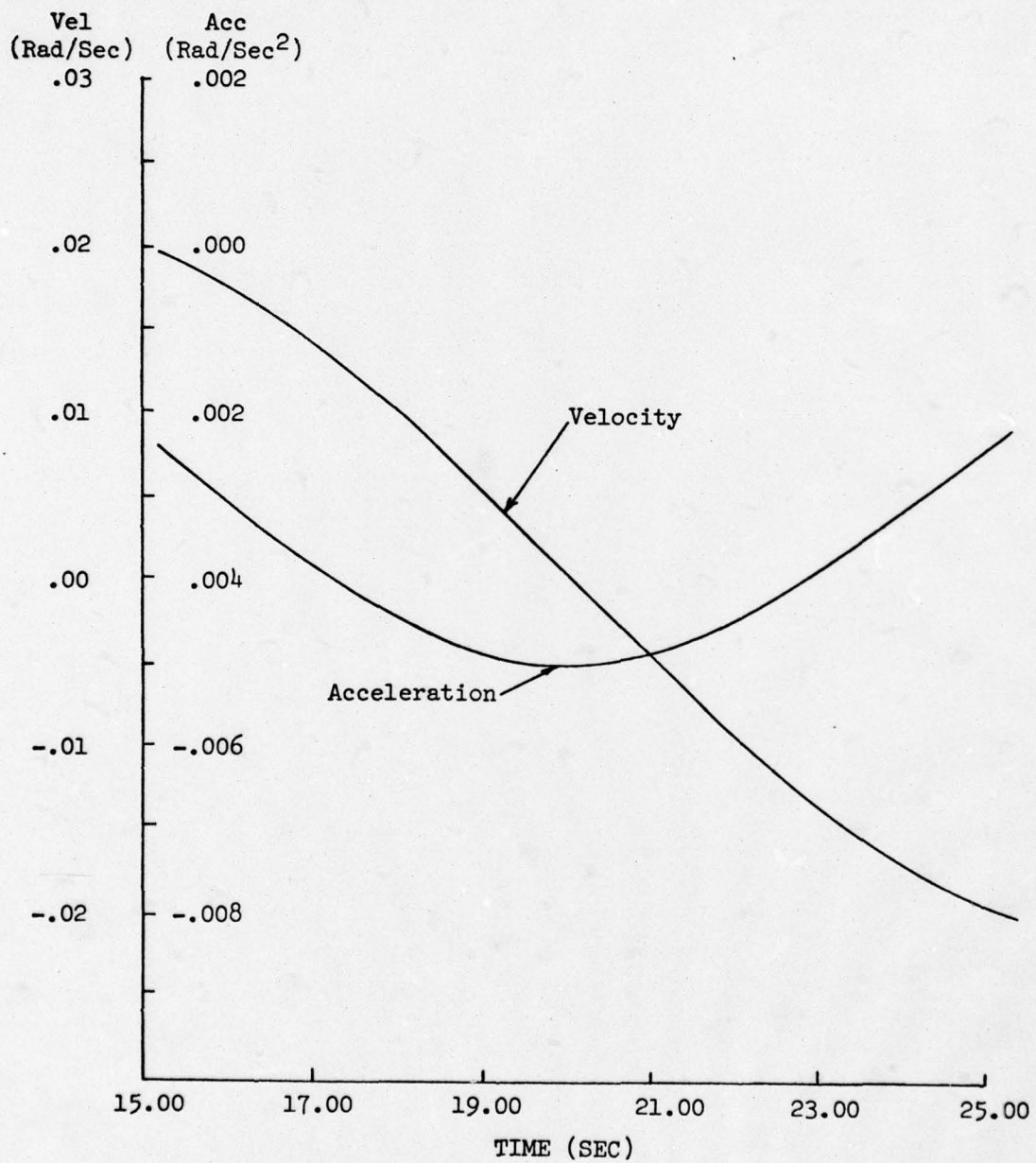
error as indicated by the simulator response in Fig. 7.3.3.
During this phase of the trajectory the average position
of the azimuth sight is also changing from a sight leading
situation to a sight lagging situation. The position of
the azimuth sight on the finite image size may have some
effect on the elevation operator positioning his cross hair.

7.3.3 Trajectory 2, Azimuth Axis

As seen in Fig. 6.7.2, trajectory 2 requires the tar-
get to maneuver more than the straight and level fly-by of
trajectory 1. This is also seen by the velocity and accel-
eration time functions, which are shown in Fig. 7.3.5 for
trajectory 2, azimuth axis. Table 7.3.3 presents a summary
of the gradient calculation results. Again the cost func-
tional is reduced at each iteration and the determinant of
$M(\phi^k)$ is nonzero at each iteration value of ϕ^k. Iteration
one used the values of ϕ arrived at for the last iteration
of the azimuth axis for trajectory 1. The most significant
change in the adjustment of ϕ during the four iterations on
the trajectory was to increase the strength of the white
noise portion of the disturbance, W_1. This supports our
previous comment that W_1 reflects the adaptive capability
of the operator. An increase in W_1 will increase the gain
of the Kalman filter in the model and result in the filter
increasing the weight of recent data. This is desirable
when target is maneuvering significantly, as is the case
for this trajectory. Of course, some of this adaptability

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FIG. 7.3.5
TRAJECTORY 2, AZIMUTH AXIS
ANGULAR VELOCITY AND ACCELERATION

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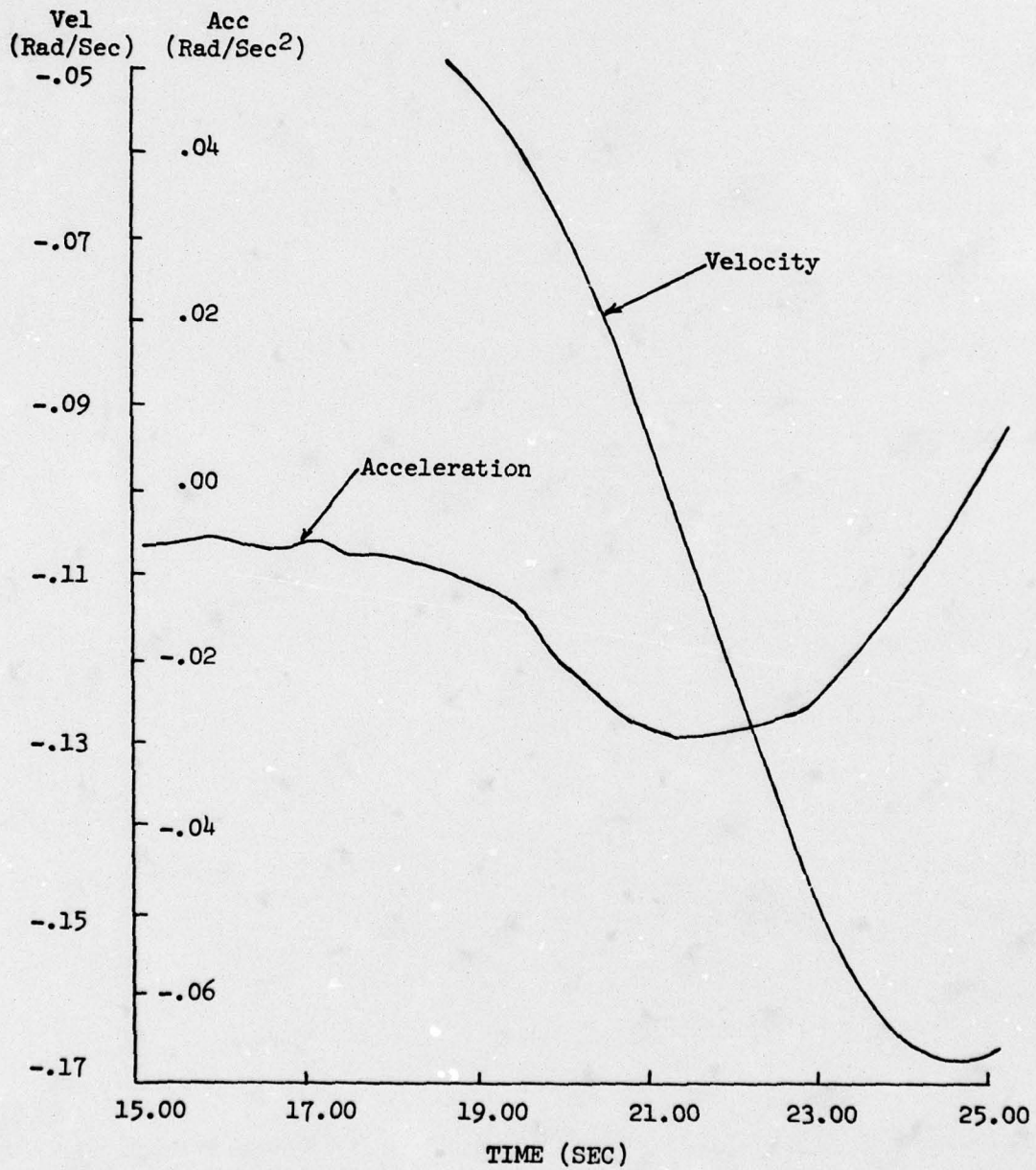
Table 7.3.3

SUMMARY OF GRADIENT ITERATIONS
TRAJECTORY 2, AZIMUTH AXIS

k=1	ϕ^k	$-\frac{1}{2}\frac{\partial J(\phi^k)}{\partial \phi^k}$	$2\left[N^k\right]^{-1}$
$J(\phi^1)=.277\times10^{-3}$	.108	$.2156\times10^{-3}$	10
	.2055	$-.2217\times10^{-3}$	10
$\|M(\phi^1)\|=.189\times10^{-12}$	.0570	$.4254\times10^{-3}$	10
	.0467	$-.1566\times10^{-3}$	10
	.0313	$.8054\times10^{-3}$	10
	.00001003	$.1151\times10^1$	10^{-5}
k=2			
$J(\phi^2)=.236\times10^{-3}$	.110	$-.1449\times10^{-3}$	10
	.2033	$-.5845\times10^{-3}$	10
$\|M(\phi^2)\|=.896\times10^{-13}$	.0621	$.1678\times10^{-4}$	10
	.0451	$-.9019\times10^{-3}$	10
	.0394	$-.7995\times10^{-4}$	10
	.00002154	.9472	10^{-5}
k=3			
$J(\phi^3)=.201\times10^{-3}$	.109	$-.2627\times10^{-4}$	1
	.1975	$-.3918\times10^{-3}$	1
$\|M(\phi^3)\|=.486\times10^{-13}$	.0623	$.9452\times10^{-4}$	1
	.0361	$-.6307\times10^{-3}$	1
	.0386	$.8722\times10^{-4}$	1
	.00003101	.7853	10^{-6}
k=4			
$J(\phi^4)=.199\times10^{-3}$	.109	$-.2367\times10^{-4}$	
	.1971	$-.3848\times10^{-3}$	
$\|M(\phi^4)\|=.471\times10^{-13}$	.0624	$.9431\times10^{-4}$	
	.0355	$-.6225\times10^{-3}$	
	.0387	$.8641\times10^{-4}$	
	.00003180	.7740	

reflected by the model also having the square of the angular acceleration, $\left[\theta_T(t)\right]^2$, contribute to the strength of the disturbance as seen by the filter.

Fig. 7.3.6 compares the model response, using parameter values after iteration four, to the simulator response. Here the mean tracking error of the model is of slightly greater magnitude than that of the simulator, although both have the sight leading the target. The larger standard deviation of the simulator between 15 and 18 seconds may be attributed to crossover effects from the elevation axis. As seen from Fig. 7.3.7, the elevation acceleration during this period is relatively high, which tends to increase the variance of the elevation tracking error. The azimuth operator may be affected by this apparent disturbance in the elevation axis with the result being an increase in the azimuth tracking error variance. One could have the model account for this by one of two methods: (1) add a term to equivalent observation noise in the azimuth axis that is proportional to variance of the observed quantities in the elevation axis (and vice versa) or (2) add a term in the variance of the plant disturbance seen by the Kalman filter in the azimuth axis that is proportional to the square of the elevation axis angular acceleration (and vice versa). Either of these modifications would provide an increase in the variance of the tracking error that was proportional to the variance in the other axis.

FIGURE 7.3.6
TRAJECTORY 2, AZIMUTH AXIS
MODEL AND SIMULATOR TRACKING ERRORS
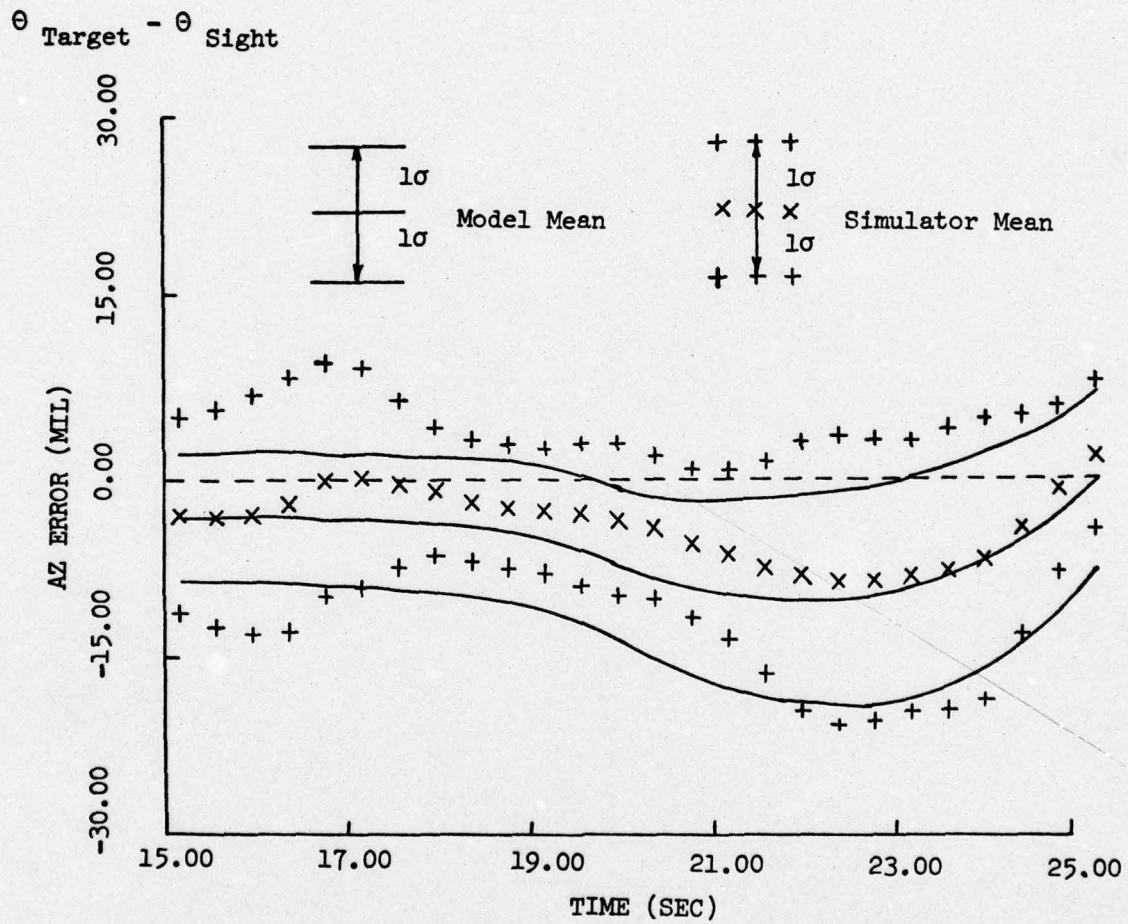
FIG. 7.3.7
TRAJECTORY 2, ELEVATION AXIS
ANGULAR VELOCITY AND ACCELERATION

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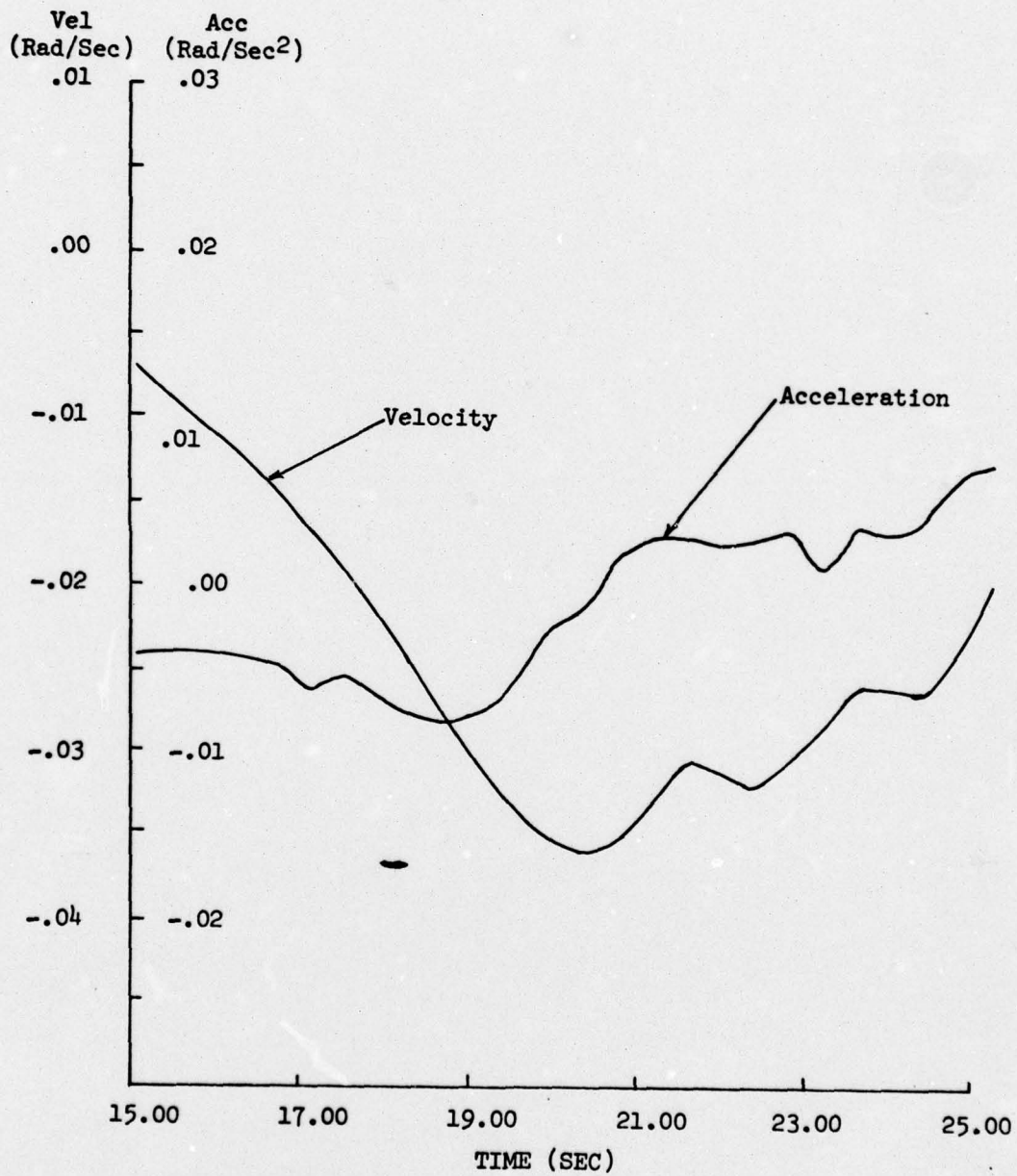
7.3.4 Trajectory 2, Elevation Axis

The elevation tracking task is also more difficult for trajectory 2 than for trajectory 1. This is exemplified by the angular velocity and acceleration time functions shown in Fig. 7.3.7. Table 7.3.4 summarizes the gradient calculations for this trajectory and, as with the other cases, the cost functional is reduced at each iteration and the determinant of $M(\phi^k)$ is nonzero, supporting local identifiability of ϕ^k. Fig. 7.3.8 compares the model response to the simulator response using the parameter values after iteration four. The model mean and standard deviation are generally in good agreement with the corresponding simulator values. The simulator mean tracking error is less accurate than that predicted by the model in the 19 to 23 second region when the sight is changing from an average leading position to one in which the sight lags the target. From Fig. 7.3.7, it is seen that this is the region in which the angular acceleration is changing from a negative to a positive value. The overshoot in the 22 second region exhibited by the simulator could be attributed to the operator anticipating the point in the trajectory where the target angular velocity is no longer increasing in magnitude. Although the operators were presented with one of three trajectories at random, it is suspected that after several trials the characteristics of the trajectories are learned to some degree. Another possible explanation for the difference is that the numerically calculated acceleration

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Table 7.3.4

SUMMARY OF GRADIENT ITERATIONS
TRAJECTORY 2, ELEVATION AXIS

k=1	ϕ^k	$-\dfrac{1}{2}\dfrac{\partial J(\phi^k)}{\partial\phi^k}$	$2\left[N^k\right]^{-1}$
$J(\phi^1)=.211\times10^{-3}$	.104	$.4029\times10^{-3}$	10
	.2000	$.2773\times10^{-3}$	10
$\|M(\phi^1)\|=.381\times10^{-15}$	.0366	$.3984\times10^{-3}$	10
	.0327	$.7430\times10^{-3}$	10
	.0214	$.9502\times10^{-3}$	10
	.00001352	.4070	10^{-5}
k=2			
$J(\phi^2)=.136\times10^{-3}$	.108	$.2816\times10^{-3}$	10
	.2028	$.2187\times10^{-3}$	10
$\|M(\phi^2)\|=.435\times10^{-15}$	.0406	$.2792\times10^{-3}$	10
	.0401	$.4623\times10^{-3}$	10
	.0309	$.6433\times10^{-3}$	10
	.00001752	.2070	10^{-5}
k=3			
$J(\phi^3)=.104\times10^{-3}$	.111	$.1816\times10^{-3}$	10
	.2049	$.1535\times10^{-3}$	10
$\|M(\phi^3)\|=.487\times10^{-15}$	.0434	$.1842\times10^{-3}$	10
	.0447	$.2929\times10^{-3}$	10
	.0373	$.3928\times10^{-3}$	10
	.00001959	.1243	10^{-5}
k=4			
$J(\phi^4)=.910\times10^{-4}$	.113	$.1148\times10^{-3}$	
	.2064	$.1021\times10^{-3}$	
$\|M(\phi^4)\|=.518\times10^{-15}$	.0452	$.1178\times10^{-3}$	
	.0476	$.1877\times10^{-3}$	
	.0412	$.2227\times10^{-3}$	
	.00002083	$.8173\times10^{-1}$	

FIGURE 7.3.8
TRAJECTORY 2, ELEVATION AXIS
MODEL AND SIMULATOR TRACKING ERRORS
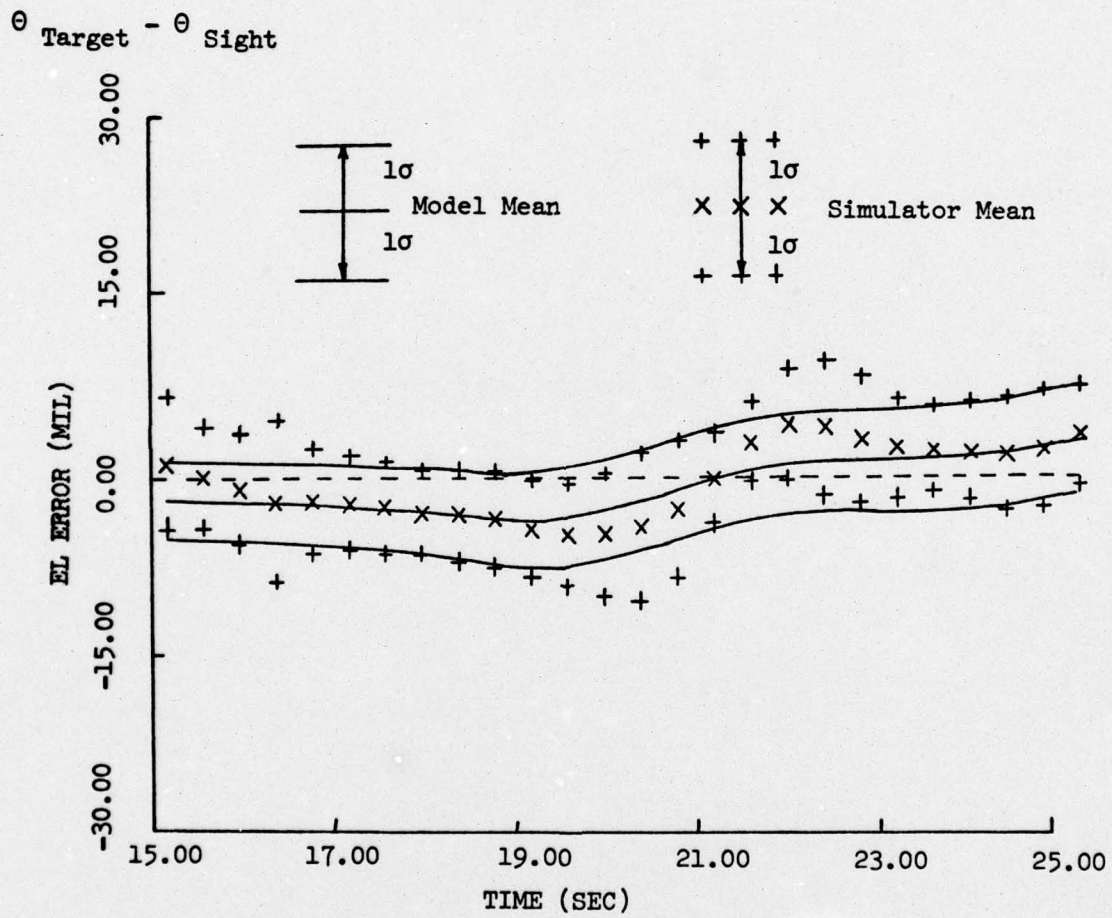
disturbance may not exactly duplicate the actual angular acceleration seen by the simulator. The possibility of some crosscoupling between the two axes could also be a factor. It is noted that during the 19 to 23 seconds, both the azimuth and elevation operators have relatively diffi-cult tracking tasks. The effect of a changing sight position in azimuth axis on the elevation operator, and vice versa, is not readily apparent; however, some cross-coupling may take place. As mentioned before, some small effects could be introduced by the changing target image on the TV monitor.

7.3.5 Alternative Model for Plant Disturbance

As discussed in Section 5.4, one could model the plant disturbance without the white noise component; that is, the disturbance would be only the deterministic quantity $w_2(t)$. In our case $w_2(t)$ is the angular acceleration. Following the approach by Kleinman (Refs 20,23), the covariance kernel of this disturbance as seen by the Kalman filter is modeled by

$$\dot{E}\{[w(t_1) - \bar{w}(t_1)]^T[w(t_2) - \bar{w}(t_2)]\} = 2 \tau_c \ddot{\Theta}(t_1)\delta(t_1 - t_2)$$

where τ_c may be viewed as a correlation time for $\ddot{\Theta}(t)$.

A second set of calculations was performed with this alternative model for the disturbance. τ_c replaced W_1 as a quantity to be identified in the parameter vector ϕ. Table 7.3.5 and Figs. 7.3.9 through 7.3.12 summarize the

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Table 7.3.5

RESULTS OF GRADIENT CALCULATIONS
ALTERNATIVE MODEL FOR PLANT DISTURBANCE

Trajectory	Final Iteration	ϕ^k	$J(\phi^k)$	$-\dfrac{1}{2}\dfrac{\partial J(\phi^k)}{\partial \phi^k}$
1, Azimuth	4	.119	$.291 \times 10^{-4}$	$-.2798 \times 10^{-4}$
		.2078		$-.2590 \times 10^{-5}$
		.0977		$-.4452 \times 10^{-5}$
		.0572		$-.1637 \times 10^{-5}$
		.0579		$-.1584 \times 10^{-4}$
		6.919		$-.1417 \times 10^{-7}$
1, Elevation	3	.110	$.309 \times 10^{-4}$	$-.5943 \times 10^{-4}$
		.2044		$-.5370 \times 10^{-4}$
		.0587		$-.3171 \times 10^{-4}$
		.0503		$-.8055 \times 10^{-4}$
		.0416		$-.8839 \times 10^{-4}$
		4.234		$-.1212 \times 10^{-5}$
2, Azimuth	4	.109	$.162 \times 10^{-3}$	$-.1410 \times 10^{-3}$
		.2026		$-.2552 \times 10^{-3}$
		.0551		$.5174 \times 10^{-4}$
		.0466		$-.3243 \times 10^{-3}$
		.0385		$.1588 \times 10^{-4}$
		1.303		$.6521 \times 10^{-4}$
2, Elevation	4	.114	$.120 \times 10^{-3}$	$.1004 \times 10^{-4}$
		.2061		$.1017 \times 10^{-3}$
		.0602		$.7093 \times 10^{-4}$
		.0531		$.1599 \times 10^{-3}$
		.0453		$.7540 \times 10^{-4}$
		5.099		$-.1672 \times 10^{-5}$

FIGURE 7.3.9
TRAJECTORY 1, AZIMUTH
MODEL AND SIMULATOR TRACKING ERRORS
(ALTERNATIVE MODEL FOR PLANT NOISE)
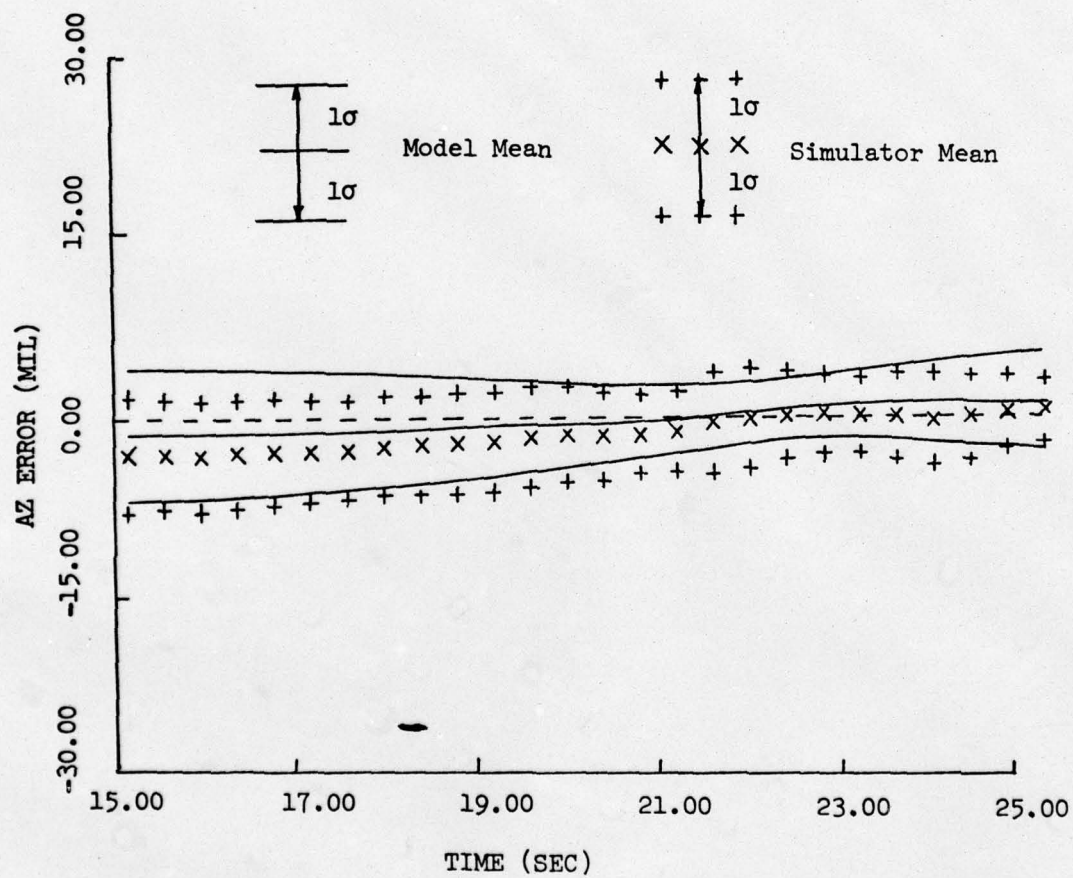
FIGURE 7.3.10
TRAJECTORY 1, ELEVATION
MODEL AND SIMULATOR TRACKING ERRORS
(ALTERNATIVE MODEL FOR PLANT NOISE)

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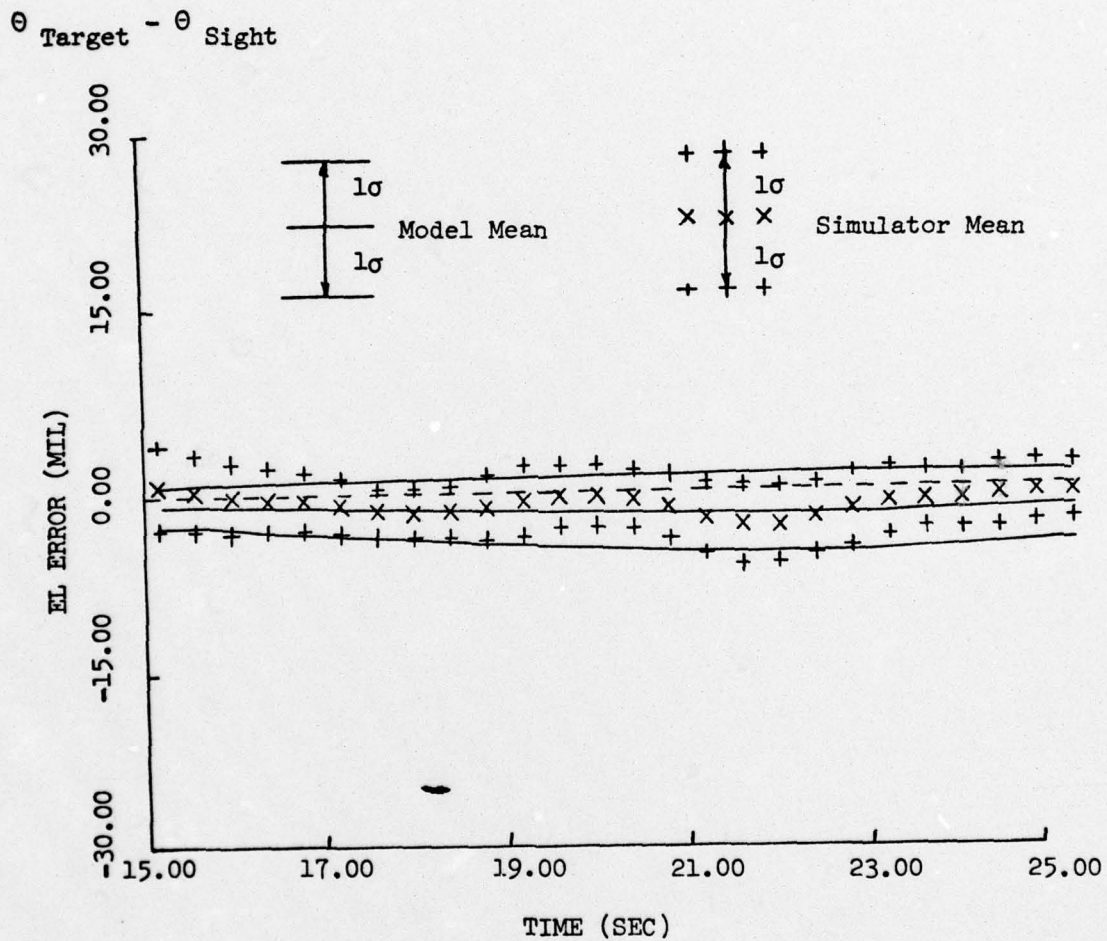
FIGURE 7.3.11
TRAJECTORY 2, AZIMUTH
MODEL AND SIMULATOR TRACKING ERRORS
(ALTERNATIVE MODEL FOR PLANT NOISE)
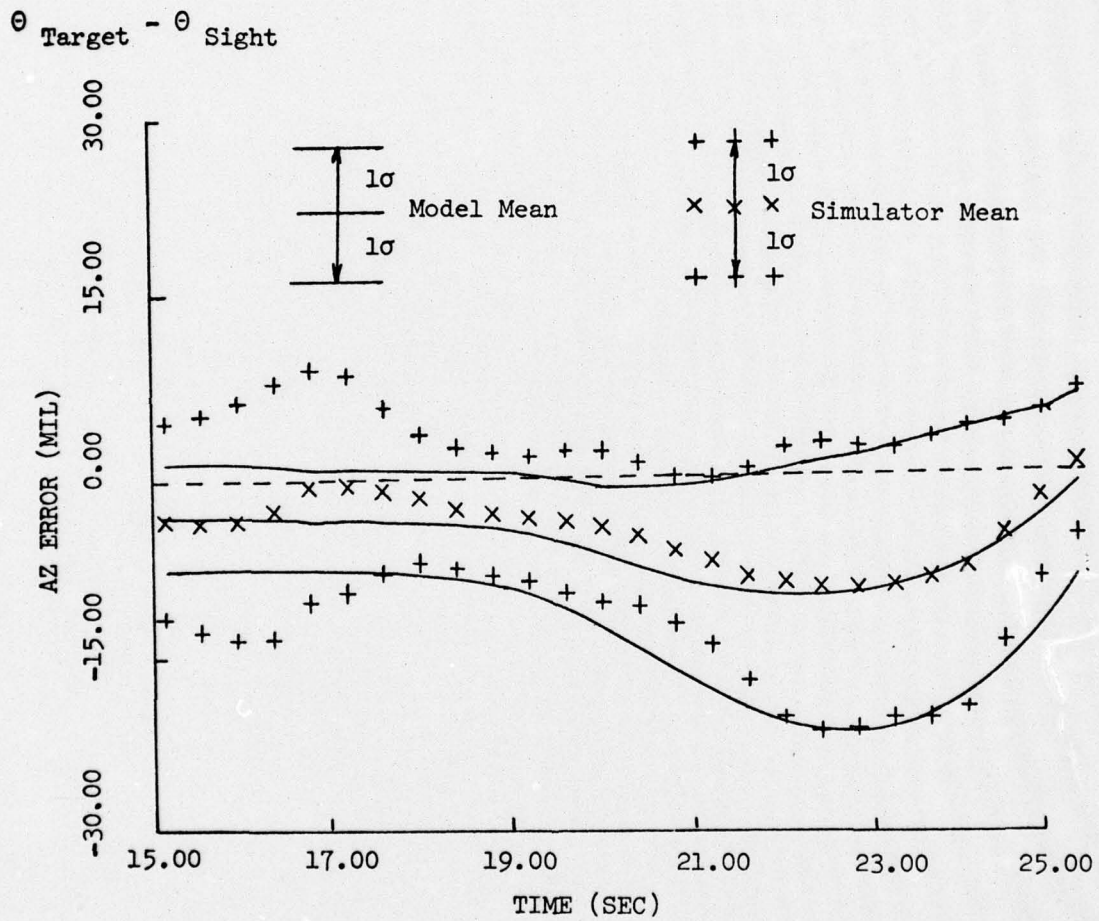
FIGURE 7.3.12
TRAJECTORY 2, ELEVATION
MODEL AND SIMULATOR TRACKING ERRORS
(ALTERNATIVE MODEL FOR PLANT NOISE)

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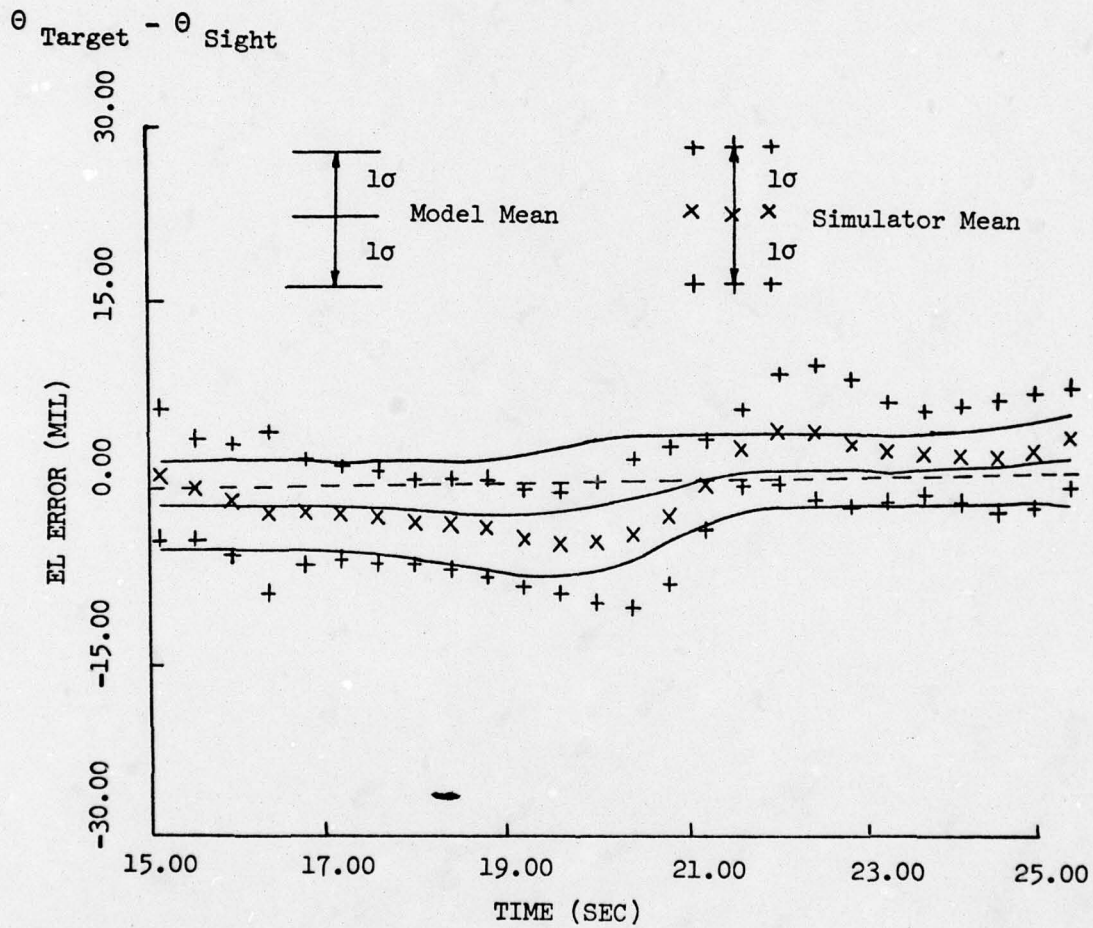
results obtained using this approach. Comparing this data to that obtained with the first model for plant noise, we see that the cost functional obtained after the same number of iterations is not significantly different in either case. A somewhat better agreement of model and simulator data is obtained for the new formulation of the plant disturbance for the elevation axis of trajectory 1 and the azimuth axis of trajectory 2, although both provide acceptable results.

A model could be formulated which includes the white noise components $W_1(t)$ in the disturbance and also maintains the factor τ_c in the variance kernel. That is

$$w(t) = W_1(t) + \ddot{\theta}^2(t)$$

and the process noise covariance kernel as seen by the Kalman filter is

$$W(t)\delta(t) = W_1\delta(t) + 2\,\tau_c\,\ddot{\theta}^2(t)\delta(t)$$

In this manner W_1 could account for input disturbances and model uncertainties which do not depend upon $\ddot{\theta}^2(t)$. The factor $2\,\tau_c\,\ddot{\theta}^2(t)$ would reflect operator adaptability throughout the trajectory as the operator perceives a changing situation, with τ_c permitting adjustment of this capability.

7.3.6 Summary

In this chapter, the identifiability of the parameters of the optimal control model was analyzed using the theory

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developed in the earlier chapters. It was found the parameters of the model are identifiable when operating in a tracking task. This is a very important result from the standpoint of the viability of the model structure, since model validation is dependent on the parameters being identifiable under some combination of input and measured output data. This combination of input signal and observation process that yields identifiability is certainly not unique. Under certain observation processes, some of the parameters may not be identifiable (Ref 40), but this should not be used as a basis for claiming unidentifiability unless it is the only available observation process. The input signal (angular acceleration) and observation process (ensemble averages and standard deviations of tracking errors for many trials) was chosen to conform to available simulator data. It was fortuitous that such a combination does permit identifiability. Although this approach differs from the more usual approach of basing identification on time histories of individual trials, it has the advantage of matching model response to the average response of the actual system over many trials and many operators. Also, one obtains, as a derived measured quantity, the standard deviation of this response, which provides additional information for the identification process. The concept of using a deterministic trajectory as the basis for the input signal (target angular acceleration) allows repeatability of the experiment over many trials and forms a basis for predicting a nonzero mean

response. As will be discussed below, the deterministic
input also would permit a systematic investigation of out-
put sensitivities to various inputs (trajectories).

For the computations that were performed, the mean
tracking error and the tracking error standard deviation
were used as the measured data. The results of Section 7.2
indicate that this observation process should provide suf-
ficient data to permit identification of model parameters.
In the computations it was decided to adjust each of the
model parameters at each iteration using the gradient
technique for obtaining the parameter change, $\Delta\phi^k$. This
resulted in the cost functional being reduced at each
iteration, thereby verifying the technique; however, fur-
ther research could test other computation techniques with
various combinations of inputs and measured data. Another
important factor is the assessment of sensitivity of output
data to the parameter adjustments for various inputs. A
measure of output sensitivity is obtained from

$$S_\ell = \sum_{i=1}^{m} \sum_{j=1}^{k} \left[\frac{\partial y_i(t_j)}{\partial \phi_\ell} \right]^2$$

where $y_i(t_j)$ is the ith component of the measured vector
$y(t_j)$ and ϕ_ℓ is the ℓth component of the parameter vector ϕ.
This sensitivity function could be evaluated for each param-
eter under various inputs and observations. The matrix

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$$M(\phi) = \sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial \phi} \right]^T \left[\frac{\partial y(t_i)}{\partial \phi} \right]$$

yields some information concerning the sensitivity of measured data to changes in ϕ for the given input signal. In the computations reported previously, we found that the determinant of $M(\phi^k)$ was largest for the trajectory 2 calculations than for trajectory 1, and the smallest value was obtained for the elevation axis of trajectory 1. This can be attributed to the relatively small angular acceleration exhibited by the target in elevation for this trajectory. Thus, the output data apparently is less sensitive to changes of ϕ for the elevation axis of this trajectory than for the other three cases. More research is warranted on the subject of input signal sensitivity.

The computational results indicate that the optimal control model does adequately predict human response in a tracking task. The mean response and standard deviation of the response of the model predictions were in general agreement with those of the simulator for each case presented. As mentioned in the previous discussion of the results, further refinements to the model are warranted. Such refinements should consider the possibility of cross-coupling between axes and methods for modeling such effects. Also the effect of the changing shape of the finite image on the display needs to be researched. In Section 6.6, a method was presented for modeling the indifference threshold

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to account for finite image size; however, this does not account for the possibility of the changing image affecting the mean response. That is, for a given trajectory, the image presented is a deterministic quantity as a function of time which may influence the operators in positioning their sights.

In the next chapter we will summarize the results of this research and recommend topics for further research.

Chapter VIII

SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

In this chapter, the major objectives and contributions
of this research will be reviewed, and then some potential
areas of extension and future research will be discussed.

8.1 Objectives and Contributions

The first objective was to develop sufficient con-
ditions for establishing observability and identifiability
of nonlinear dynamical systems. We considered systems
described by

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \qquad (8.1.1)$$

for all $t \in [t_o, t_f]$ and an associated observation process
of the form

$$y(t_i) = h(t_i, g(t_i, x_o, u)) \qquad i = 1, 2, \cdot \cdot \cdot k$$

$$(8.1.2)$$

where $x(t)$ is an n vector, $y(t_i)$ is an m vector with
$t_i \in [t_o, t_f]$ for all i, u is an r vector valued function
of t, and $g(t, x_o, u)$ is a solution of Eq (8.1.1) for all
$t \in [t_o, t_f]$. The observability problem is concerned with
determining conditions under which knowledge of the input
and observed output data uniquely determine the state of
the system. The approach taken was to view the observation

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sequence $\{y(t_i)\}$ as an mk-vector

$$Y = \left[y^T(t_1)y^T(t_2) \cdot \cdot \cdot y^T(t_k)\right]^T \qquad (8.1.3)$$

Then the nonlinear system is <u>observable</u> under the observation process Y with the input u if there is a one-to-one correspondence between the initial conditions x_o and the observation vector $Y(x_o)$. The approach allowed us to use the inverse function theorem as a basis for stating sufficient conditions for local observability under discrete observation processes in terms of the rank of the Jacobian matrix $\partial Y(x_o)/\partial x_o$ (see Theorem 2.2). An alternate condition based on Theorem 2.2, but using the properties of the Gram matrix, requires the nonsingularity of

$$M(\phi) = \sum_{i=1}^{k}\left[\frac{\partial y(t_i)}{\partial x_o}\right]^T\left[\frac{\partial y(t_i)}{\partial x_o}\right]$$

for local observability (see Theorem 2.3). Application of these theorems require a minimum of $k = n/m$ observation times and, in general, an a priori knowledge of the solution so that an expression can be obtained for $y(t_i) = h(t_i,x_o,u))$. An alternative test which avoids these problems is derived based on an nm-vector $\bar{F}(t, x)$ where

$$\bar{F}(t, x) = \left[F_o^T(t, x)F_1^T(t, x) \cdot \cdot \cdot F_{n-1}^T(t, x)\right]^T \qquad (8.1.4)$$

and $F_i(t, x)$ given by the recursion relation Eq (2.1.14). Using the previous results in conjunction with the recursion relation, sufficient conditions for local observability are

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stated in terms of the rank of $\partial \bar{F}(t, x)/\partial x$ evaluated at $t^* \epsilon \{t_i\}$ (see Theorem 2.6). These results are extended to systems with continuous observation processes in Theorems 2.4, 2.5, and 2.7.

Sufficient conditions for global observability are obtained for an open, path-connected set by adding to the conditions for local observability the requirement of norm-coerciveness or continuous differentiability (see Theorem 2.11). Alternative sufficient conditions for global observability are established using the properties of strictly monotone and convex functions (see Theorems 2.14 and 2.16). These results are combined with the recursion relation of Eq (2.1.14) to give a very useful observability test (see Theorem 2.18). Again these results are extended to continuous time measurements in Theorems 2.17 and 2.19.

The results summarized above provide new and useful conditions for establishing observability of nonlinear dynamical systems. As shown in Section 2.4, the theory which is developed can also have useful applications to linear systems.

In Chapter III, the problem of identifying parameters of nonlinear dynamical systems is addressed. Here we considered systems described by

$$\dot{x}(t) = f(t, x(t), u(t), \phi) \qquad x(t_o) = x_o \qquad (8.1.5)$$

for all $t \epsilon [t_o, t_f]$, where ϕ is a vector of s parameters. The parameter vector ϕ of the nonlinear system is

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identifiable under the observation process with input u and initial conditions x_0 if there is a one-to-one correspondence between the parameter vector ϕ and the observation vector $Y(\phi)$. By viewing parameter identifiability as a special case of observability, sufficient conditions for identifiability are established based on the development in Chapter II. Theorems 3.1, 3.2, and 3.3 state sufficient conditions for local identifiability for discrete and continuous observations based on the rank of $\partial Y/\partial \phi$ and the nonsingularity of $\sum_{i=1}^{k} \left[\partial y(t_i)/\partial \phi\right]^T \left[\partial y(t_i)/\partial \phi\right]$. The recursion relation of Eq (2.1.14) is used to develop alternative sufficient conditions for local identifiability (see Theorem 3.4). Again, using the development in Chapter II, sufficient conditions for global identifiability on an open, path-connected set are established by adding to the requirements for local identifiability continuous differentiability or norm-coerciveness. Also global identifiability is established based on strict monotinicity relative to ϕ of the observations (see Theorems 3.6, 3.7, and 3.8). Stochastic identifiability is investigated for the case of additive zero-mean white noise on the observation. It was found that deterministic identifiability was both necessary and sufficient for stochastic identifiability for the additive noise case when independent repetitions of the observation process can be obtained (see Section 3.5).

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The theory developed in Chapter III provides new tests for establishing identifiability of nonlinear dynamical systems. These results are considered significant in view of the fundamental nature of the identifiability question in conjunction with system modeling and because in many situations the state equations are not linear with respect to the parameter vector ϕ. As exemplified by Section 3.4, these developments have useful applications to linear systems, as well.

The second major objective of this research was to use the theory developed for identifying nonlinear dynamical systems to investigate the identifiability of the optimal control model for human operators. The model equations were developed to conform to a simulator where the measured data consists of ensemble averages and variances of many trials of tracking data using the simulator. We found that each of the parameters are globally identifiable on $(0, \infty)$ from observations of the mean and standard deviation of the tracking errors or of the mean and standard deviation of the manual control inputs to the tracking system (see Section 7.2). This is a significant result because the fundamental characteristic of parameter identifiability had not been established for this model, even though it had been successful in simulating human response. Further significance is attached to this result since it exemplifies an application of the theory to a nontrivial nonlinear dynamical identification problem. An additional result

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concerning the local identification of the model parameters was obtained from the computational results, where we found that for each case the matrix

$$M(\phi) = \sum_{i=1}^{k} \left[\frac{\partial y(t_i)}{\partial \phi} \right]^T \left[\frac{\partial y(t_i)}{\partial \phi} \right]$$

was nonsingular (see Section 7.1). From Theorem 3.2, this implies the parameter vector ϕ is locally identifiable.

Computations were performed using the gradient technique to find an estimate of the model parameter vector ϕ which minimizes the cost functional

$$J = \sum_{i=1}^{k} \left[z(t_i) - y(t_i) \right]^T \left[z(t_i) - y(t_i) \right]$$

where the measured data was the azimuth and elevation mean tracking errors and standard deviations (see Section 7.3). Each axis was treated as an independent case since the system had two people in the tracking system: one operated the azimuth control and one the elevation control. Starting with nominal values for the parameters, the computational technique was successful in reducing the cost functional with each iteration in adjusting the parameter values. These results are considered significant since they establish the feasibility of using a computational technique in conjunction with a cost functional for estimating the model parameters, as opposed to previous work where model parameters were modified in a heuristic fashion to fit measured data.

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A third objective was to provide further insight into the modeling of human response by the optimal control model. Although the modeling approach was basically that used by Kleinman (Refs 20,23), there were several unique aspects. System dynamics, sensor displays, and control mechanisms differ from previous studies; particularly the use of two person controls. In addition to these unique aspects, some significant aspects of the model which were illuminated by this research include:

(1) An apparent operator indifference threshold due to the finite image size of the target on the monitor which increases the magnitude of the observation noise covariance when tracking within the threshold (see Section 6.6). An additional effect of the changing image shape on the mean tracking response is suggested by comparing model and simulator responses (see Section 7.3).

(2) The results support the model configuration which treats target angular acceleration as a plant disturbance (see Section 5.4 and 7.3). In addition, the results suggest some cross coupling between axes which could be modeled by adding an equivalent observation noise in the azimuth axis that is proportional to the variance of the observed quantities in the elevation axis, and vice versa (see Section 7.3).

The above findings, together with the general agreement of the model tracking response with the simulator tracking

response, complement previous research concerning the optimal control model for simulating human response.

8.2 Possible Extensions and Future Research

There are several aspects of the nonlinear observability and identification problem which warrant additional effort. An area mentioned, but not explored in detail in this research, is the effect of the input function on the observability and identification problem. There are two aspects of this phase of the problem which could be researched more thoroughly. One is the effect of the input function on the binary question of whether a system is observable or identifiable. The second area concerns the effect of the input function on the sensitivity of output data to changes in initial conditions or system parameters. As mentioned in Chapter VII, a measure of output sensitivity is obtained by evaluating

$$S_\ell = \sum_{i=1}^{m} \sum_{j=1}^{k} \left[\frac{\partial y_i(t_j)}{\partial \phi_\ell} \right]^2$$

where $y_i(t_j)$ is the ith component of the measured vector $y(t_j)$ and ϕ_ℓ is the ℓth component of the parameter vector ϕ. The effect of the input signal on this sensitivity function needs to be researched for nonlinear systems.

In Chapter III, a special case of stochastic identifiability was investigated for nonlinear dynamical systems with additive zero-mean white noise on the observations.

There is need for further effort on this difficult problem of stochastic identifiability of nonlinear systems.

In Chapters II and III, the emphasis was placed on establishing sufficient conditions for observability and identifiability of nonlinear systems. A better understanding of the problem would result from a formulation of both the necessary and sufficient conditions. Also, some additional effort would be useful in relating the classical theory of nonlinear differential equations to the problem of observability and identifiability. For example, how do the requirements for observability affect the properties of the solution to the system differential equation?

There are areas requiring further effort related to the identification of the optimal control model parameters. Although the identifiability of the optimal control model parameters was established by the research reported in this document under a given combination of input/output data, additional research needs to be done. For example, the model was placed in the context of a tracking task in this research to conform with an available simulator. The identifiability of the model parameters in other applications, such as pilot modeling, and under other input/output combinations needs to be investigated.

The parameter sensitivity question needs to be addressed in conjunction with identifying the parameters of the optimal control model. This is an important aspect of the problem of

determining accurate estimates of the parameter values, once identifiability has been established. As mentioned previously, although the gradient technique used in this research appeared to be working satisfactorily, additional effort is warranted to investigate the effectiveness of other computational techniques. Additional computational work needs to be done in conjunction with possible model refinements to estimate the model parameter values more confidently.

Additional analysis is required to assess the need for further model refinements. Further effort is needed with respect to cross coupling between axes and target image effects on operator threshold and biasing. Another area needing investigation is the operator cost functional; for example, during the acquisition phase, where tracking errors are large, the mean squared error cost functional probably does not apply.

This research establishes the basis for addressing the areas outlined above. Clearly, there is a need for considerably more research concerning the properties of nonlinear dynamical systems. Although the nonlinear aspect can slow the progress of the researcher, the reward can be great because of the extreme importance of this class of systems.

Appendix A

FRECHET AND GATEAUX DERIVATIVES

Some basic properties of Frechet and Gateaux derivatives of n-dimensional functions will be reviewed in this section (see Ref 39:59).

Definition: A mapping $F : D \subset R^n \to R^m$ is <u>Gateaux-differentiable</u> at an interior point x of D if there exists a linear operator $A \in L(R^n, R^m)$ such that, for any $h \in R^n$

$$\lim_{t \to 0} (1/t) \, || \, F(x + th) - F(x) - t \, A(h) \, || = 0 \, .$$

If F is Gateaux differentiable, the linear operator A is unique, is denoted by F'(x), and is called the G-derivative at x.

Definition: A mapping $F : D \subset R^n \to R^m$ is <u>Frechet-differentiable</u> at $x \in int(D)$ if there is an $A \in L(R^n, R^m)$ such that

$$\lim_{h \to 0} (1/||h||) \, || \, F(x + h) - F(x) - A(h) \, || = 0 \, .$$

If F is Frechet differentiable, the linear operator A is unique, is denoted by F'(x), and is called the F-derivative at x.

Definition: The F-derivative of $F : D \subset R^n \to R^m$ at $x_0 \in D$ is <u>strong</u> if, given any $\epsilon > 0$, there is a $\delta > 0$ so that the set $\bar{S}(x_0, \delta) \overset{\Delta}{=} \{y \in R^n \ni || \, y - x_0 \, || < \delta\}$ is a subset of D

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and

$$\| F(y) - F(x) - F'(x_o)(y - x) \| \leq \epsilon \| y - x \|$$

for all $x, y \in \bar{S}(x_o, \delta)$.

A.1

Suppose that F has an F-derivative at each point of an open neighborhood of $x_o \in D$. Then F' is strong at x_o if and only if F' is continuous at x.

A.2

F can have a strong derivative at x_o even though F is not differentiable at all points of an open neighborhood of x_o.

A.3

If $F : D \subset R^n \rightarrow R^m$ is F-differentiable at x, then F is continuous at x. That is, there is a $\delta > 0$ and a $C \geq 0$ such that

$$\| F(x + h) - F(x) \| \leq C \|h\|$$

whenever $\|h\| \leq \delta$.

A.4

F is G-differentiable at x if it is F-differentiable at x.

A.5

$F'(x)$ is continuous at x if and only if all the partial derivatives $\partial f_i / \partial x_j$ are continuous at x.

A.6

If F has a G-derivative at each point of an open neighborhood of x and F'(x) is continuous at x, then F is F-differentiable in an open neighborhood of x and the F-derivative is continuous.

A.7

If F has a continuous G-derivative on the open set $D_0 \subset D$, we will say F is <u>continuously</u> <u>differentiable</u> on D_0.

Appendix B

CONDITIONS FOR UNIQUE SOLUTIONS OF DIFFERENTIAL SYSTEMS

In this appendix, we will briefly review some conditions which will assure that the system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o$$

for all $t \in [t_o, t_f]$ has a unique solution (denoted by $g(t, x, u)$) on $[t_o, t_f]$ (see Ref 47).

<u>Definition</u>: A mapping $F : D \subset R^n \to R^m$ is <u>Holder-continuous</u> on $D_o \subset D$ if there exist constants $C \geq 0$ and $p \in (0, 1]$ so that for all $x, y \in D$,

$$|| F(y) - F(x) || \leq C || y - x ||^p.$$

If $p = 1$, then F is <u>Lipschitz-continuous</u> on D_o.

B.1

Given the system

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(t_o) = x_o \qquad \text{(B.1)}$$

Let $f(\cdot, \cdot, \cdot)$ be piecewise continuous in the first argument, Lipschitz-continuous in the second argument, and continuous in the third argument for all $t \in [t_o, t_f]$. Then given any t_o, x_o and piecewise continuous $u(\cdot)$, the system differential Eq (B.1) has one and only one solution $g(t, x_o, u)$ on $[t_o, t_f]$. Thus for any $t_1 \in [t_o, t_f]$, the solution $g(t_1, x_o, u)$ is uniquely determined by t_o, x_o, and $u_{[t_o, t_1]}$.

B.2

If $\partial f/\partial x$ is continuous for all x and all $t \in [t_o, t_f]$ and $\| x(t) \| < \infty$, then $g(t, x_o, u)$ has a derivative in a neighborhood of x_o which is continuous at x_o.

Appendix C

EXISTENCE OF SOLUTION TO RICCATI EQUATION

We want to show that our formulation of the linear regulator problem results in the existence of a solution to the steady state Riccati equation.

Consider the linear equation

$$\dot{x}(t) = A\, x(t) + B\, u(t) \qquad (C.1)$$

with observations

$$z(t) = D\, x(t) \qquad (C.2)$$

and a cost functional

$$J = \int_{o}^{t_f} \left[z^T(t) Q\, z(t) + g\, u^2(t) \right] dt \qquad (C.3)$$

with Q positive definite and $g > 0$.

The associated Riccati equation for the linear regulator is

$$-\dot{P}(t) = P(t)A + A^T P(t) - P(t)B\, g^{-1}\, B^T P(t) + D^T Q\, D \qquad (C.4)$$

Kwakernaak and Sivan (Ref 10) have stated conditions for the Riccati equation to have a steady state solution. As $t_f \rightarrow \infty$, the solution of the Riccati equation approaches a constant steady state value if and only if the system as defined by Eqs (C.1) and (C.2) possesses no poles that are at the same time unstable, uncontrollable, and reconstructable

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In the context of the optimal control model in Chapter V, we have

$$A = \begin{bmatrix} A_n & 0 & 0 \\ \hline A_b & A_d & B_c \\ \hline 0 & 0 & 0 \end{bmatrix}$$

$A_n = \ell \times \ell$

$A_b = n \times \ell$

$A_d = n \times n$

$B_c = n \times 1$

$$B = \begin{bmatrix} 0 \\ \hline 0 \\ \hline 1 \end{bmatrix}$$

$\ell \times 1$

$n \times 1$

1×1

$$D = \begin{bmatrix} 0 & D_1 & 0 \end{bmatrix}$$

$D = (\ell + n + 1) \times (\ell + n + 1)$

$D_1 = (\ell + n + 1) \times n$

$$Q = \begin{bmatrix} \ell \times \ell & & \\ 0 & 0 & 0 \\ \hline 0 & Q_d & 0 \\ \hline 0 & 0 & 0 \\ & & 1 \times 1 \end{bmatrix}$$

$Q_d = n \times n$

With the above forms for the matrices A, B, D, and Q it is clear that the first ℓ states (disturbance states) are unstable and uncontrollable. However, to be reconstructable, the present state $x(t)$ must be determined from knowledge of past outputs $\{z(\sigma) = D\, x(\sigma) : \sigma \leq t\}$. From the form of D, it is clear that the first ℓ states are not reconstructable.

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Although the states of the dynamics (states $\ell + 1$ through $\ell + n$) may be reconstructable, they are also controllable. Thus our formulation of the linear regulator problem leads to the existence of a solution to the steady state Riccati equation.

Appendix D

EFFECT OF HOLDING NEURO-MUSCULAR DELAY, τ_n, CONSTANT ON SOLUTION OF RICCATI EQUATION

In Section 5.3, it was discussed that there is physical motivation to hold the neuro-muscular delay, τ_n, constant. We want to show that for the class of problems under consideration, specifying a value of τ_n completely determines the optimal control gains, λ.

Recall from Eq (5.3.9) that the optimal gains are obtained by

$$\lambda = \frac{1}{g} \left[P_{13} \mid P_{23} \mid P_{33} \right] \tag{D.1}$$

Let $\ell_{13} = \frac{P_{13}}{P_{33}}$ and $\ell_{23} = \frac{P_{23}}{P_{33}}$

so that

$$\lambda = \frac{P_{33}}{g} \left[\ell_{13} \mid \ell_{23} \mid 1 \right] \tag{D.2}$$

From Eq (5.30) we know

$$\tau_n = \frac{g}{P_{33}} \tag{D.3}$$

Thus Eq (D.2) becomes

$$\lambda = \frac{1}{\tau_n} \left[\ell_{13} \mid \ell_{23} \mid 1 \right] \tag{D.4}$$

We will show below that for the cost functionals with weighting matrix

$$Q_d = \begin{bmatrix} 0 & | & 0 \\ -- & | & -- \\ 0 & | & q \end{bmatrix}$$

ℓ_{13} and ℓ_{23} can be obtained independent of the value of q. That is, the value of τ_n completely specifies λ.

With τ_n constant, the partitioned Riccati equations (5.3.5), (5.3.6), and (5.3.7) become

$$\tau_n \ell_{22} A_d + \tau_n A_d^T \ell_{22} + \frac{Q_d}{P_{33}} - \ell_{23} \ell_{23}^T = 0 \qquad (D.5)$$

$$\ell_{22} B_c + A_d^T \ell_{23} - \frac{1}{\tau_n} \ell_{23} = 0 \qquad (D.6)$$

$$2 \ell_{23}^T B_c - \frac{1}{\tau_n} = 0 \qquad (D.7)$$

where

$$\ell_{22} = \frac{P_{22}}{P_{33}} \quad \text{and} \quad \ell_{23} = \frac{P_{23}}{P_{33}} .$$

Note that ℓ_{22} is an n x n symmetric matrix and ℓ_{23} is a n vector. Let Q_d have the form

$$Q_d = \begin{bmatrix} 0 & | & 0 \\ -- & | & -- \\ 0 & | & q \end{bmatrix}$$

With this form, the factor q appears in only one of the above system of $[n(n+1)/2] + n + 1$ equations. If we ignore the equation involving q, then the system represents

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$[n(n+1)/2]$ + n equations with the same number of unknowns
(i.e., $n(n+1)/2$ components of ℓ_{22} and n components of ℓ_{23}).
Thus we can solve for the values of ℓ_{22} and ℓ_{23} independent
of the value of q. The equation where q appears determines
the value of P_{33} in terms of ℓ_{22}, ℓ_{23}, and q.

Now define $\ell_{11} = P_{11}/P_{33}$, $\ell_{12} = P_{12}/P_{33}$, and
$\ell_{13} = P_{13}/P_{33}$. Then equations (5.3.2), (5.3.3), and (5.3.4)
become

$$\tau_n \ell_{11} A_n + \tau_n \ell_{12} A_b + \tau_n A_n^T \ell_{11} + \tau_n A_b^T \ell_{12}^T - \ell_{13} \ell_{13}^T = 0$$

$$\tag{D.8}$$

$$\tau_n \ell_{12} A_d + \tau_n A_n^T \ell_{12} + \tau_n A_b^T \ell_{22} - \ell_{13} \ell_{23}^T \approx 0 \tag{D.9}$$

$$\ell_{12} B_c + A_n^T \ell_{13} + A_b^T \ell_{22} - \tau_n \ell_{13} = 0 \tag{D.10}$$

With ℓ_{22} and ℓ_{23} known, the above system represents
$[\ell(\ell+1)/2 + (\ell \times n) + \ell]$ equations and the same number of
unknowns.

From the above, it is apparent that for a specified
value of τ_n we can solve the partitioned Riccati equation
for ℓ_{13} and ℓ_{23} independent of the value of q. Then

$$\lambda = \frac{1}{\tau_n} \left[\ell_{13} \mid \ell_{23} \mid 1 \right] . \tag{D.11}$$

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Appendix E

SIMULATOR RICCATI EQUATION SOLUTION

Let $P_{13}{}^T \stackrel{\Delta}{=} y_1$

$\quad\quad P_{23}{}^T \stackrel{\Delta}{=} [y_2 \; y_3 \; y_4]$

$\quad\quad P_{33} \stackrel{\Delta}{=} y_5$

$\quad\quad P_{12} \stackrel{\Delta}{=} [x_{12} \; x_{13} \; x_{14}]$

$$P_{22} \stackrel{\Delta}{=} \begin{bmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Eqs (6.4.5) through (6.4.10) become:

$$[x_{12} \; x_{13} \; x_{14}] \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + [0 \; 0 \; -1] \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} - \frac{1}{g} y_1^2 = 0 \tag{E.1}$$

$$[x_{12} \; x_{13} \; x_{14}] \begin{bmatrix} 0 & 1 & 0 \\ -\alpha_1 & -\beta_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + [0 \; 0 \; -1] \begin{bmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$- \frac{1}{g} y_1 [y_2 \; y_3 \; y_4] = [0 \; 0 \; 0] \tag{E.2}$$

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$$\begin{bmatrix} x_{12} & x_{13} & x_{14} \end{bmatrix} \begin{bmatrix} 0 \\ K_2 \cos\Theta_E \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_2 & y_3 & y_4 \end{bmatrix}$$

$$- \frac{1}{g} y_1 y_5 = 0 \qquad (E.3)$$

$$\begin{bmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\alpha_1 & -\beta_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_1 & 1 \\ 1 & -\beta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{bmatrix} - \frac{1}{g} \begin{bmatrix} y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (E.4)$$

$$\begin{bmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{bmatrix} \begin{bmatrix} 0 \\ K_2 \cos\Theta_E \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_1 & 1 \\ 1 & -\beta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$- \frac{1}{g} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} y_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (E.5)$$

$$2 \begin{bmatrix} y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 0 \\ K_2 \cos\Theta_E \\ 0 \end{bmatrix} - \frac{y_5^2}{g} = 0 \qquad (E.6)$$

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Eqs (E.4), (E.5), and (E.6) can be solved for the controllable state gains. Expanding Eqs (E.4), (E.5), and (E.6), one can obtain the following set of equations:

$$2(-\alpha_1 x_{32} + x_{42}) = \frac{1}{g} y_2^2 \qquad (E.7)$$

$$x_{42} - \beta_1 x_{32} = \frac{1}{g} y_4 y_3 \qquad (E.8)$$

$$2(x_{23} - \beta_1 x_{33}) = \frac{1}{g} y_3^2 \qquad (E.9)$$

$$\gamma_2 x_{33} + y_2 - \beta_1 y_3 - \frac{y_5}{g} y_3 = 0 \qquad (E.10)$$

$$2\gamma_2 y_3 - \frac{y_5^2}{g} = 0 \qquad (E.11)$$

$$x_{22} - \beta_1 x_{23} - \alpha_1 x_{33} + x_{43} = \frac{y_2 y_3}{g} \qquad (E.12)$$

$$\gamma_2 x_{23} - \alpha_1 y_3 + y_4 - \frac{y_5 y_2}{g} = 0 \qquad (E.13)$$

$$\gamma_2 x_{43} - \frac{y_5 y_4}{g} = 0 \qquad (E.14)$$

$$\frac{1}{g} y_4^2 = q \qquad (E.15)$$

where $\gamma_2 = K_2 \cos\theta_E$.

The above nine equations have nine unknowns which can be solved as outlined below.

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From these one can show that

$$y_4 = \sqrt{g\,q} \tag{E.16}$$

$$y_5 = \sqrt{2\,g\,\gamma_2\,y_3} \tag{E.17}$$

$$y_2 = \frac{1}{2A_3}\left[1 - \sqrt{1 - 4\,A_3 E_1}\right] \tag{E.18}$$

where

$$E_1 = A_4 y_3^2 + A_5 y_3 + A_6 y_5 + \frac{y_5 y_3}{g_1} \tag{E.19}$$

$$A_3 y_2^2 + \frac{y_5 y_2}{\beta_1 g} + \left[\frac{-\gamma_2 y_4}{\alpha_1 g \beta_1} + \frac{\alpha_1}{\beta_1}\right] y_3 - A_2 y_5 - \frac{y_4}{\beta_1} = 0 \tag{E.20}$$

where

$$A_2 = \frac{y_4}{\alpha_1 g} \qquad\qquad A_4 = \frac{\gamma_2}{2\beta_1 g}$$

$$A_3 = \frac{\gamma_2}{2\alpha_1 \beta_1 g} \qquad\qquad A_5 = \frac{-A_2 \gamma_2}{\beta_1} + \beta_1$$

$$A_6 = -A_2$$

The above equations can be solved for y_2, y_3, and y_5 as follows. Note that y_4 is determined from Eq (E.16). First solve

$$1 - 4\,A_3 E_1 = 0 \tag{E.21}$$

for maximum value of y_3.

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Note that Eq (E.21) is only a function of y_3 when Eq (E.17) and Eq (E.19) are combined. This is solved using the Newton iteration method.

Let

$$F_3 = E_1 - \frac{1}{4A_3} = 0$$

$$= A_4 y_3^2 + A_5 y_3 + A_6 y_5 + \frac{y_5 y_4}{g} - \frac{1}{4A_3} \qquad (E.22)$$

$$\frac{dF_3}{dy_3} = 2A_4 y_3 + A_5 + \frac{A_6 g \, \gamma_2}{y_5} + \frac{\gamma_2}{y_5} y_3 + \frac{y_5}{g} \qquad (E.25)$$

$$\triangleq F_3{}'$$

Using the Newton method

$$y_3^{n+1} = y_3^n - F_3(y_3^n)/F_3{}'(y_3^n) \qquad (E.24)$$

This gives the maximum value for y_3, denoted as $y_3{}^*$; then use $y_3{}^* - \varepsilon$ as the initial value for solving Eq (E.20). That is

$$F_4 = A_3 y_2^2 + \frac{y_5 y_2}{g\beta_1} + \left[\frac{-\gamma_2 y_4}{\beta_1 \alpha_1 g} + \frac{\alpha_1}{\beta_1}\right] y_4 - A_2 y_5 - \frac{y_5}{\beta_1} = 0$$

$$(E.25)$$

$$F_4{}' = \frac{dF_4}{dy_3} = 2A_3 y_2 \frac{dy_2}{dy_3} + \frac{y_5}{g_1 \beta_1} \frac{dy_2}{dy_3} + \left[\frac{-\gamma_2 y_4}{\beta_1 \alpha_1 g} + \frac{\alpha_1}{\beta_1}\right]$$

$$+ \left[-A_2 + \frac{y_2}{g\beta_1}\right] \frac{g\gamma_2}{y_5} \qquad (E.26)$$

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where

$$\frac{dy_2}{dy_3} = \frac{1}{\sqrt{1-4A_3E_1}} \frac{dF_3}{dy_3} \qquad \text{(E.27)}$$

Using the initial value $y_3 = y_3{}^* - \varepsilon$, one solves

$$y_3{}^{n+1} = y_3{}^n - F_4/F_4' \qquad \text{(E.28)}$$

This gives values for y_2, y_3, y_4, and y_5. To obtain y, the following logic can be employed.

From Eq (E.2) one obtains

$$- x_{44} = \frac{1}{g} y_1 y_4 \qquad \text{(E.29)}$$

From Eq (E.4) one obtains

$$- \alpha_1 x_{34} + x_{44} = \frac{1}{g} y_2 y_4 \qquad \text{(E.30)}$$

From Eq (E.14)

$$x_{34} = \frac{\gamma_2 y_5 y_4}{g} \qquad \text{(E.31)}$$

Combining Eqs (E.29), (E.30), and (E.31) we get for y_1

$$y_1 = - y_2 - \frac{\alpha_1 y_5}{\gamma_2} \qquad \text{(E.32)}$$

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Appendix F

GRADIENT EQUATIONS WITH RESPECT TO PARAMETER VECTOR ϕ

F.1 Gradient with Respect to τ_n

$$\frac{\partial x_m}{\partial \tau_n} = \frac{\partial \hat{x}_{am}}{\partial \tau_n} + \tau \frac{\partial A_u}{\partial \tau_n} e^{A_u \tau} e_{1m} + e^{A_u \tau} \frac{\partial e_{1m}}{\partial \tau_n} + \frac{\partial e_{2m}}{\partial \tau_n} \qquad (F.1)$$

$$\frac{\partial \dot{\hat{x}}_{am}}{\partial \tau_n} = \frac{\partial A_c}{\partial \tau_n} \hat{x}_{am} + A_c \frac{\partial \hat{x}_{am}}{\partial \tau_n} + \frac{\partial K}{\partial \tau_n} C_a e_{1m} + K C_a \frac{\partial e_{1m}}{\partial \tau_n} \qquad (F.2)$$

$$A_c = A_a - B_a \lambda_c \qquad (F.3)$$

$$\frac{\partial A_c}{\partial \tau_n} = \frac{\partial A_a}{\partial \tau_n} - B_a \frac{\partial \lambda_c}{\partial \tau_n} - \frac{\partial B_a}{\partial \tau_n} \lambda_c \qquad (F.4)$$

$$\frac{\partial A_a}{\partial \tau_n} = \begin{bmatrix} 0 & 0 \\ 0 & (1/\tau_n)^2 \end{bmatrix} \qquad (F.5)$$

$$\frac{\partial B_a}{\partial \tau_n} = \begin{bmatrix} 0 \\ -1/\tau_n{}^2 \end{bmatrix} \qquad (F.6)$$

$$\frac{\partial \lambda_c}{\partial \tau_n} = \frac{\partial \lambda_c}{\partial g} \frac{\partial g}{\partial \tau_n} \qquad (F.7)$$

From Eq (5.3.9)

$$\lambda_{ci} = \tau_n \lambda_i = \frac{\tau_n}{g} y_i \qquad \text{(see Appendix E)}$$

$$\text{for } i = 1, 2, 3, 4 \tag{F.8}$$

and

$$\lambda_5 = \frac{1}{\tau_n} = \frac{1}{g} y_5 = \frac{1}{g} \sqrt{2 g \gamma_2 y_3} \tag{F.9}$$

Thus $g = \tau_n y_5$ where $y_5 = \sqrt{2 g \gamma_2 y_3}$ (F.10)

$$\frac{\partial g}{\partial \tau_n} = y_5 + \frac{\partial y_5}{\partial g} \tau_n$$

$$= y_5 + \tau_n \left[\frac{\gamma_2 y_3}{P_{22}} + \frac{\gamma_2 g}{P_{22}} \frac{\partial y_3}{\partial g} \right] \tag{F.11}$$

From Eqs (F.8) and (F.9)

$$\lambda_{ci} = \frac{1}{y_5} y_i \qquad i = 1, 2, 3, 4 \tag{F.12}$$

$$\frac{\partial \lambda_{ci}}{\partial \tau_n} = \frac{1}{y_5} \frac{\partial y_i}{\partial g} - \frac{1}{y_5^3} \left[\gamma_2 y_3 + \gamma_2 g \frac{\partial y_3}{\partial y} \right] y_i \tag{F.13}$$

To evaluate $\dfrac{\partial y_i}{\partial g}$, we use the equations developed in Appendix E for y_i. These are:

$$A_3 y_2^2 + \frac{y_5}{\beta_1 g} y_2 + \left[\frac{-\gamma_2 y_5}{\alpha_1 g \beta_1} + \frac{\alpha_1}{\beta_1} \right] y_3 - A_2 y_5 - \frac{y_4}{\beta_1} = 0 \tag{F.14}$$

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where

$$A_3 = \gamma_2/2\alpha_1 g\beta_1 \qquad\qquad A_5 = \frac{-\gamma_2 y_4}{g\alpha_1\beta_1} + \beta_1$$

$$A_2 = y_4/g\alpha_1 \qquad\qquad A_6 = -\frac{y_4}{g\alpha_1}$$

$$y_5 = \sqrt{2\,\gamma_2 g y_3} \qquad\qquad A_4 = \frac{\gamma_2}{2\beta_1 g}$$

$$y_4 = \sqrt{q\,g} \qquad\qquad\qquad (F.15)$$

$$y_2 = \frac{1}{2A_3}\left[1 - \sqrt{1-4A_3 E_1}\right] \qquad (F.16)$$

where

$$E_1 = A_4 y_3^2 + A_5 y_3 + A_6 y_5 + \frac{y_5 y_3}{g}$$

$$y_1 = -y_2 - \left[\frac{\alpha_1}{\gamma_2}\right] y_5 \qquad\qquad (F.17)$$

Taking the partial of Eq (F.14) with respect to g yields

$$2\,A_3 y_2 \frac{\partial y_2}{\partial g} - \frac{\gamma_2}{2\,\alpha_1 g^2 \beta_1} y_2^2 + \frac{y_5}{\beta_1 g}\frac{\partial y_2}{\partial g} + \frac{y_2}{\beta_1 g}\frac{\partial y_5}{\partial g}$$

$$-\frac{y_5 y_2}{\beta_1 g^2} + \left[\frac{-\gamma_2 y_5}{\alpha_1 g\beta_1} + \frac{\alpha_1}{\beta_1}\right]\frac{\partial y_3}{\partial g} - \frac{\gamma_2 y_3}{\alpha_1 g\beta_1}\frac{\partial y_4}{\partial g} + \frac{\gamma_2 y_4 y_3}{\alpha_1 g^2 \beta_1}$$

$$- A_2 \frac{\partial y_5}{\partial g} - \frac{y_5}{g\alpha_1}\frac{\partial y_4}{\partial g} + \frac{y_4 y_5}{g^2 \alpha_1} - \frac{1}{\beta}\frac{\partial y_4}{\partial g} = 0 \qquad (F.18)$$

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$$Y_5 = \sqrt{2\,\gamma_2 g Y_3}$$

$$\frac{\partial Y_5}{\partial g} = \frac{\gamma_2 Y_3}{Y_5} + \frac{\gamma_2 g}{Y_5}\,\frac{\partial Y_3}{\partial g} \tag{F.19}$$

$$Y_4 = \sqrt{g\,q}$$

$$\frac{\partial Y_4}{\partial g} = \frac{q}{2Y_4} \tag{F.20}$$

Substituting into Eq (F.18) and rearranging

$$\left[2\,A_3 Y_2 + \frac{Y_5}{\beta_1 g} \right]\frac{\partial Y_2}{\partial g} + \left[\frac{Y_2}{\beta_1 g}\,\frac{\gamma_2 g}{Y_5} + \left(\frac{-\gamma_2 Y_4}{\alpha_1 g \beta_1} + \frac{\alpha_1}{\beta_1} \right) \right.$$

$$\left. -\,A_2\,\frac{\gamma_2 g}{Y_5} \right]\frac{\partial Y_3}{\partial g} = \frac{\gamma_2}{2\,\alpha_1 g \beta_1}\,Y_2{}^2 - \frac{Y_2 \gamma_2 Y_3}{\beta_1 q\,Y_5} + \frac{P_{22} Y_2}{\beta_1 g^2}$$

$$+\,\frac{\gamma_2 Y_3 q}{\alpha_1 g \beta_1{}^2 Y_4} - \frac{\gamma_2 Y_4 Y_3}{\alpha_1 g^2 \beta_1} + A_2\,\frac{\gamma_2 Y_3}{Y_5}$$

$$+\,\frac{Y_5 q}{2 q \alpha_1 Y_5} - \frac{Y_4 Y_5}{g^2 \alpha_1} + \frac{q}{2\beta_1 Y_4} \tag{F.21}$$

From Eq (F.16)

$$\frac{\partial Y_2}{\partial g} = \frac{1}{2A_3{}^2}\left(\frac{\gamma_2}{2\alpha_1 g^2 \beta_1} \right)\left(1 - \sqrt{1 - 4A_3 E_1} \right)$$

$$+\,\frac{1}{2A_3}\left(\frac{-1}{2\,\sqrt{1 - 4A_3 E_1}} \right)\left(-4A_3\,\frac{\partial E_1}{\partial g} + 4E_1\,\frac{\gamma_2}{2\,\alpha_1 g^2 \beta_1} \right) \tag{F.22}$$

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or

$$\frac{\partial y_2}{\partial g} = \left[\frac{E_1}{A_3 \sqrt{1-4A_3E_1}} - \frac{1}{2A_3^2} \left[1 - \sqrt{1-4A_3E_1} \right] \right] \left[\frac{\gamma_2}{2 \alpha_1 g^2 \beta_1} \right]$$

$$+ \frac{1}{\sqrt{1-4A_3E_1}} \quad \frac{\partial E_1}{\partial g} \tag{F.23}$$

From Eq (F.16)

$$\frac{\partial E_1}{\partial g} = 2 A_4 y_3 \frac{\partial y_3}{\partial g} - \frac{\gamma_2 y_3^2}{2\beta_1 g^2} + A_5 \frac{\partial y_3}{\partial g} + \frac{\gamma_2 y_4 y_3}{g^2 \alpha_1 \beta_1} - \frac{y_3 \gamma_2}{g \alpha_1 \beta_1} \frac{\partial y_4}{\partial g}$$

$$+ A_6 \frac{\partial y_5}{\partial g} + \frac{y_5 y_4}{g^2 \alpha_1} - \frac{y_5}{g \alpha_1} \frac{\partial y_4}{\partial g} + \frac{y_5}{g} \frac{\partial y_3}{\partial g} - \frac{y_5 y_3}{g^2}$$

$$+ \frac{y_3}{g} \frac{\partial y_5}{\partial g} \tag{F.24}$$

or

$$\frac{\partial E_1}{\partial g} = 2 A_4 y_3 + A_5 + A_6 \frac{\gamma_2 g}{y_5} + \frac{y_5}{g} + \frac{y_3 \gamma_2}{y_5} \quad \frac{\partial y_3}{\partial g} - \frac{\gamma_2 y_3^2}{2\beta_1 g^2}$$

$$+ \frac{\gamma_2 y_4 y_3}{g^2 \alpha_1 \beta_1} - \frac{y_3 \gamma_2 g}{g \alpha_1 \beta_1 2 y_4} + A_6 \frac{\gamma_2 y_3}{y_5} + \frac{y_5 y_4}{g^2 \alpha_1}$$

$$- \frac{y_5 g}{g \alpha_1 2 y_4} - \frac{y_5 y_3}{g^2} + \frac{y_3^2 \gamma_2}{g y_5} \tag{F.25}$$

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From Eqs (F.23) and (F.25) we get

$$\frac{\partial y_2}{\partial g} = A_{52} + A_{53}\frac{\partial y_3}{\partial g} \qquad\qquad (F.26)$$

with

$$A_{52} = \frac{E_1}{A_3\sqrt{1-4A_3E_1}} - \frac{1}{2A_3^2}\left[1 - \sqrt{1-4A_3E_1}\right]\left[\frac{\gamma_2}{2\alpha_1 g^2\beta_1}\right]$$

$$+ \frac{1}{\sqrt{1-4A_3E_1}}\left[\frac{-\gamma_2 y_3^2}{2\beta_1 g^2} + \frac{\gamma_2 y_4 y_3}{g^2\alpha_1\beta_1} - \frac{y_3\gamma_2 q}{2\alpha_1\beta_1 g y_4}\right.$$

$$\left. + A_6\frac{\gamma_2 y_3}{y_5} + \frac{y_5 y_4}{g^2\alpha_1} - \frac{y_5 q}{2g\alpha_1 y_4} - \frac{y_5 y_3}{g^2} + \frac{\gamma_2 y_3^2}{y_5 g}\right]$$

and

$$A_{53} = \frac{1}{\sqrt{1-4A_3E_1}}\left[2\,A_4 y_3 + A_5 + A_6\frac{\gamma_2 g}{y_5} + \frac{y_5}{g} + \frac{y_3\gamma_2}{y_5}\right]$$

Substituting Eq (F.26) into Eq (F.21) allows evaluation of $\frac{\partial y_3}{\partial g}$ in terms of known quantities.

From Eq (F.17)

$$\frac{\partial y_1}{\partial g} = -\frac{\partial y_2}{\partial g} - \frac{\alpha_1}{\gamma_2}\frac{\partial y_5}{\partial g} \qquad\qquad (F.27)$$

where $\frac{\partial y_2}{\partial g}$ is given by Eq (F.26) and $\frac{\partial y_5}{\partial g}$ is given by Eq (F.19).

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The above allows evaluation of $\dfrac{\partial A_c}{\partial \tau_n}$.

$$K = e^{A_a \tau} G \tag{F.28}$$

$$\frac{\partial K}{\partial \tau_n} = \tau \frac{\partial A_a}{\partial \tau_n} e^{A_u \tau} G + e^{A_u \tau} \frac{\partial G}{\partial \tau_n} \tag{F.29}$$

$$G = P_1 C_a^{\ T} V^{-1} \tag{F.30}$$

$$\frac{\partial G}{\partial \tau_n} = C_a^{\ T} V^{-1} \frac{\partial P_1}{\partial \tau_n} \tag{F.31}$$

From the Eq (2.3.10) for propagating P_1 we obtain

$$\frac{\partial \dot{P}_1}{\partial \tau_n} = \frac{\partial A_a}{\partial \tau_n} P_1 + A_a \frac{\partial P_1}{\partial \tau_n} + P_1 \frac{\partial A_a^{\ T}}{\partial \tau_n} + \frac{\partial P_1}{\partial \tau_n} A_a^{\ T}$$

$$- \frac{\partial P_1}{\partial \tau_n} C_a^{\ T} V^{-1} C_a P_1 - P_1 C_a^{\ T} V^{-1} C_a \frac{\partial P_1}{\partial \tau_n}$$

$$+ \frac{\partial \Gamma_a}{\partial \tau_n} W \Gamma_a^{\ T} + \Gamma_a W \frac{\partial \Gamma_a^{\ T}}{\partial \tau_n} \tag{F.32}$$

$$\frac{\partial \Gamma_a}{\partial \tau_n} = \begin{bmatrix} 0 & 0 \\ 0 & -1/\tau_n^2 \end{bmatrix} \tag{F.33}$$

From Eq (5.5.2) we get

$$\frac{\partial \dot{e}_{1m}}{\partial \tau_n} = \frac{\partial A_f}{\partial \tau_n} e_{1m} + A_f \frac{\partial e_{1m}}{\partial \tau_n} \tag{F.34}$$

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$$A_f = A_a - G\, C_a$$

$$\frac{\partial A_f}{\partial \tau_n} = \frac{\partial A_a}{\partial \tau_n} - \frac{\partial G}{\partial \tau_n}\, C_a \tag{F.35}$$

$$\frac{\partial e_2}{\partial \tau_n} = \left[\int_o^\tau \frac{\partial A_a}{\partial \tau_n}\, \varepsilon\, e^{A_a \varepsilon}\, d\varepsilon \right] \Gamma\, w_2(\sigma) \tag{F.36}$$

The above equations allows calculation of $\frac{\partial x_{am}}{\partial \tau_n}$ as a function of time using Eq (F.1).

To calculate $\frac{\partial P_a}{\partial \tau_n}$ we refer to Eqs (5.4.27), (5.4.29), (5.4.30), (5.4.31), and (5.4.32).

$$\frac{\partial P_a}{\partial \tau_n} = \tau\, \frac{\partial A_a}{\partial \tau_n}\, e^{A_a \tau}\, P_2 e^{A_a^T \tau} + e^{A_a \tau}\, \frac{\partial P_2}{\partial \tau_n}\, e^{A_a^T \tau}$$

$$+ e^{A_a \tau}\, P_2\, \tau\, \frac{\partial A_a^T}{\partial \tau_n}\, e^{A_a^T \tau} + \frac{\partial P_3}{\partial \tau_n}\, e^{A_a^T \tau}$$

$$+ P_3 \tau\, \frac{\partial A_a^T}{\partial \tau_n}\, e^{A_a^T \tau} + \tau\, \frac{\partial A_a}{\partial \tau_n}\, e^{A_a \tau}\, P_3^T$$

$$+ e^{A_a \tau}\, \frac{\partial P_3^T}{\partial \tau_n} + \frac{\partial P_4}{\partial \tau_n} + \frac{\partial P_5}{\partial \tau_n} \tag{F.37}$$

with

$$\frac{\partial \dot{P}_2}{\partial \tau_n} = \frac{\partial A_f}{\partial \tau_n}\, P_2 + A_f\, \frac{\partial P_2}{\partial \tau_n} + \frac{\partial P_2}{\partial \tau_n}\, A_f^T + P_2\, \frac{\partial A_f^T}{\partial \tau_n} + \frac{\partial G}{\partial \tau_n}\, VG^T$$

$$+ G\, V\, \frac{\partial G^T}{\partial \tau_n} - \frac{2V_m}{\tau_n^3} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{F.38}$$

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$$\frac{\partial \dot{P}_3}{\partial \tau_n} = \frac{\partial A_c}{\partial \tau_n} P_3 + A_c \frac{\partial P_3}{\partial \tau_n} + \frac{\partial P_3}{\partial \tau_n} A_f^T + P_3 \frac{\partial A_f^T}{\partial \tau_n}$$

$$+ \frac{\partial K}{\partial \tau_n} C_a \left[P_2 - P_1 \right] + K C_a \left[\frac{\partial P_2}{\partial \tau_n} - \frac{\partial P_1}{\partial \tau_n} \right] \quad \text{(F.39)}$$

$$\frac{\partial \dot{P}_4}{\partial \tau_n} = \frac{\partial A_c}{\partial \tau_n} P_4 + A_c \frac{\partial P_4}{\partial \tau_n} + \frac{\partial P_4}{\partial \tau_n} A_c^T + P_4 \frac{\partial A_c^T}{\partial \tau_n} + \frac{\partial K}{\partial \tau_n} C_a P_3^T$$

$$+ K C_a \frac{\partial P_3^T}{\partial \tau_n} + \frac{\partial P_3}{\partial \tau_n} C_a^T K^T + P_3 C_a^T \frac{\partial K^T}{\partial \tau_n}$$

$$+ \frac{\partial K}{\partial \tau_n} V K^T + K V \frac{\partial K^T}{\partial \tau_n} \quad \text{(F.40)}$$

$$\frac{\partial P_5}{\partial \tau_n} = \int_o^\tau \epsilon \frac{\partial A_a}{\partial \tau_n} e^{A_a \epsilon} \Gamma_a \begin{bmatrix} W_1 & 0 \\ 0 & V_m \end{bmatrix} \Gamma_a^T e^{A_a^T \epsilon} d\epsilon$$

$$- \frac{2 V_m}{\tau_n^3} \int_o^\tau e^{A_a \epsilon} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} e^{A_a^T \epsilon} d\epsilon$$

$$+ \int_o^\tau e^{A_a \epsilon} \Gamma_a \begin{bmatrix} W_1 & 0 \\ 0 & V_m \end{bmatrix} \Gamma_a^T \epsilon \frac{\partial A_a^T}{\partial \tau_n} e^{A_a^T \epsilon} d\epsilon \quad \text{(F.41)}$$

The above relations allow computing $\frac{\partial P_a}{\partial \tau_n}$ as a function of time.

F.2 Gradient with Respect to τ

$$\frac{\partial x_{am}}{\partial \tau} = \frac{\partial \hat{x}_{am}}{\partial \tau} + A_a e^{A_a \tau} e_{1m} + \frac{\partial e_{2m}}{\partial \tau} \quad \text{(F.42)}$$

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$$\frac{\partial \hat{\dot{x}}_{am}}{\partial \tau} = A_c \frac{\partial \hat{x}_{am}}{\partial \tau} + \frac{\partial K}{\partial \tau} C_a e_{1m} \qquad (F.43)$$

$$\frac{\partial K}{\partial \tau} = A_a K \qquad (F.44)$$

$$\frac{\partial e_{2m}}{\partial \tau} = e^{A_a \tau} \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \ddot{\theta}_T \qquad (F.45)$$

$$\frac{\partial P_a}{\partial \tau} = A_a e^{A_a \tau} P_2 e^{A_a^T \tau} + e^{A_a \tau} P_2 A_a^T e^{A_a^T \tau}$$
$$+ P_3 A_a^T e^{A_a^T \tau} + A_a e^{A_a \tau} P_3^T + \frac{\partial P_3}{\partial \tau} e^{A_a^T \tau}$$
$$+ e^{A_a \tau} \frac{\partial P_3^T}{\partial \tau} + \frac{\partial P_4}{\partial \tau} + \frac{\partial P_5}{\partial \tau} \qquad (F.46)$$

$$\frac{\partial \dot{P}_3}{\partial \tau} = A_c \frac{\partial P_3}{\partial \tau} + \frac{\partial P_3}{\partial \tau} A_f^T + A_a K C_a [P_2 - P_1] \qquad (F.47)$$

$$\frac{\partial \dot{P}_4}{\partial \tau} = A_c \frac{\partial P_4}{\partial \tau} + \frac{\partial P_4}{\partial \tau} A_c^T + K C_a \frac{\partial P_3^T}{\partial \tau} + \frac{\partial P_3}{\partial \tau} C_a^T K^T$$
$$+ A_a K C_a P_3^T + P_3 C_a^T G^T A_a^T e^{A_u^T \tau} + A_a K V K^T$$
$$+ K V G^T A_a^T e^{A_a^T \tau} \qquad (F.48)$$

$$\frac{\partial P_5}{\partial \tau} \cong e^{A_a \tau} \Gamma_a \begin{bmatrix} W_1 & 0 \\ 0 & V_m \end{bmatrix} \Gamma_a^T e^{A_a^T \tau} \qquad (F.49)$$

F.3 Gradient with Respect to ρ_{oi}

Recall that

$$y(t) = C_a \, x_a(t) + v(t)$$

where $v(t)$ is observation noise with covariance

$$V(t) = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} C_a \, P(t) C_a^{\,T} = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$$

In the present case the operator observes angle error and angle error rate.

$$\frac{\partial x_{am}}{\partial \rho_i} = \frac{\partial \hat{x}_{am}}{\partial \rho_i} + e^{A_a \tau} \frac{\partial e_{1m}}{\partial \rho_i} \tag{F.50}$$

$$\frac{\partial \dot{\hat{x}}_{am}}{\partial \rho_i} = \frac{\partial K}{\partial \rho_i} C_a \, e_{1m} + K \, C_a \frac{\partial e_{1m}}{\partial \rho_i} + A_c \frac{\partial \hat{x}_{am}}{\partial \rho_i} \tag{F.51}$$

$$\frac{\partial K}{\partial \rho_i} = e^{A_a \tau} \frac{\partial G}{\partial \rho_i} \tag{F.52}$$

$$G = P_1 \, C_a^{\,T} \, V^{-1} \tag{F.53}$$

$$\frac{\partial G}{\partial \rho_i} = \frac{\partial P_1}{\partial \rho_i} C_a^{\,T} \, V^{-1} + P_1 \, C_a^{\,T} \frac{\partial V^{-1}}{\partial \rho_i} \tag{F.54}$$

$$\frac{\partial \dot{P}_1}{\partial \rho_i} = A_a \frac{\partial P_1}{\partial \rho_i} + \frac{\partial P_1}{\partial \rho_i} A_a^{\,T} - P_1 \, C_a^{\,T} \frac{\partial V^{-1}}{\partial \rho_i} C_a \, P_1$$

$$- \frac{\partial P_1}{\partial \rho_i} C_a^{\,T} \, V^{-1} \, C_a \, P_1 - P_1 \, C_a^{\,T} \, V^{-1} \, C_a \frac{\partial P_1}{\partial \rho_i} \tag{F.55}$$

$$\frac{\partial \dot{e}_{1m}}{\partial \rho_i} = \frac{\partial A_f}{\partial \rho_i} e_{1m} + A_f \frac{\partial e_{1m}}{\partial \rho_i} \qquad (F.56)$$

$$\frac{\partial A_f}{\partial \rho_i} = - \frac{\partial G}{\partial \rho_i} C_a \qquad (F.57)$$

$$\frac{\partial P_a}{\partial \rho_i} = e^{A_a \tau} \frac{\partial P_2}{\partial \rho_i} e^{A_a^T \tau} + \frac{\partial P_3}{\partial \rho_i} e^{A_a^T \tau} + e^{A_a \tau} \frac{\partial P_3^T}{\partial \rho_i} + \frac{\partial P_4}{\partial \rho_i} \qquad (F.58)$$

$$\frac{\partial \dot{P}_2}{\partial \rho_i} = \frac{\partial A_f}{\partial \rho_i} P_2 + A_f \frac{\partial P_2}{\partial \rho_i} + P_2 \frac{\partial A_f^T}{\partial \rho_i} + \frac{\partial P_2}{\partial \rho_i} A_f^T + \frac{\partial G}{\partial \rho_i} V G^T$$

$$+ G \frac{\partial V}{\partial \rho_i} G^T + G V \frac{\partial G^T}{\partial \rho_i} \qquad (F.59)$$

$$\frac{\partial \dot{P}_3}{\partial \rho_i} = A_c \frac{\partial P_3}{\partial \rho_i} + P_3 \frac{\partial A_f^T}{\partial \rho_i} + \frac{\partial P_3}{\partial \rho_i} A_f^T + \frac{\partial K}{\partial \rho_i} C_a \left[P_2 - P_1\right]$$

$$- K C_a \frac{\partial P_1}{\partial \rho_i} + K C_a \frac{\partial P_2}{\partial \rho_i} \qquad (F.60)$$

$$\frac{\partial \dot{P}_4}{\partial \rho_i} = A_c \frac{\partial P_4}{\partial \rho_i} + \frac{\partial P_4}{\partial \rho_i} A_c^T + \frac{\partial K}{\partial \rho_i} C_a P_3^T + K C_a \frac{\partial P_3^T}{\partial \rho_i}$$

$$+ \frac{\partial P_3}{\partial \rho_i} C_a^T K^T + P_3 C_a^T \frac{\partial K^T}{\partial \rho_i} + \frac{\partial K}{\partial \rho_i} V K^T$$

$$+ K \frac{\partial V}{\partial \rho_i} K^T + K V \frac{\partial K^T}{\partial \rho_i} \qquad (F.61)$$

$$\frac{\partial V}{\partial \rho_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} C_a P(t) C_a^T = \begin{bmatrix} V_1/\rho_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (F.62)$$

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$$\frac{\partial V}{\partial \rho_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} C_a P(t) C_a^{\ T} = \begin{bmatrix} 0 & 0 \\ 0 & V_2/\rho_2 \end{bmatrix} \qquad (F.63)$$

$$\frac{\partial V^{-1}}{\partial \rho_1} = \begin{bmatrix} -\dfrac{1}{V_1 \rho_1} & 0 \\ 0 & 0 \end{bmatrix} \qquad (F.64)$$

$$\frac{\partial V^{-1}}{\partial \rho_2} = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{V_2 \rho_2} \end{bmatrix}$$

F.4 Gradient with Respect to ρ_m

Recall that a motor noise $v_m(t)$ is added to the commanded control u_c with a covariance

$$V_m = \rho_m \ E\{u^2(t)\} \qquad (F.65)$$

where ρ_m is a constant.

$$\frac{\partial x_{am}}{\partial \rho_m} = \frac{\partial \hat{x}_{am}}{\partial \rho_m} + e^{A_a \tau} \frac{\partial e_{1m}}{\partial \rho_m} \qquad (F.66)$$

$$\frac{\partial \dot{\hat{x}}_{am}}{\partial \rho_m} = A_c \frac{\partial \hat{x}_m}{\partial \rho_m} + \frac{\partial K}{\partial \rho_m} C e_{1m} + K C \frac{\partial e_{1m}}{\partial \rho_m} \qquad (F.67)$$

$$\frac{\partial \dot{e}_{1m}}{\partial \rho_m} = \frac{\partial A_f}{\partial \rho_m} e_{1m} + A_f \frac{\partial e_1}{\partial \rho_m} \qquad (F.68)$$

$$\frac{\partial K}{\partial \rho_m} = e^{A_a \tau} \frac{\partial G}{\partial \rho_m} \qquad (F.69)$$

$$\frac{\partial G}{\partial \rho_m} = \frac{\partial P_1}{\partial \rho_m} C_a^{\ T} V^{-1} \qquad (F.70)$$

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$$\frac{\partial \dot{P}_1}{\partial \rho_m} = A_a \frac{\partial P_1}{\partial \rho_m} + \frac{\partial P_1}{\partial \rho_m} A_a{}^T + \Gamma_a \frac{\partial W_a}{\partial \rho_m} \Gamma_a{}^T$$

$$- \frac{\partial P_1}{\partial \rho_m} C_a{}^T V^{-1} C_a P_1 - P_1 C_a{}^T V^{-1} C_a \frac{\partial P_1}{\partial \rho_m} \qquad \text{(F.71)}$$

$$\frac{\partial W}{\partial \rho_m} = \begin{bmatrix} 0 & 0 \\ & \\ 0 & \partial V_m / \partial \rho_m \end{bmatrix} \qquad \text{(F.72)}$$

$$\frac{\partial P_a}{\partial \rho_m} = e^{A_a \tau} \frac{\partial P_2}{\partial \rho_m} e^{A_a{}^T \tau} + \frac{\partial P_3}{\partial \rho_m} e^{A_a{}^T \tau} + e^{A_a \tau} \frac{\partial P_3{}^T}{\partial \rho_m}$$

$$+ \frac{\partial P_4}{\partial \rho_m} + \frac{\partial P_5}{\partial \rho_m} \qquad \text{(F.73)}$$

$$\frac{\partial \dot{P}_2}{\partial \rho_m} = \frac{\partial A_f}{\partial \rho_m} P_2 + A_f \frac{\partial P_2}{\partial \rho_m} + \frac{\partial P_2}{\partial \rho_m} A_f{}^T + P_2 \frac{\partial A_f{}^T}{\partial \rho_m}$$

$$+ \frac{\partial G}{\partial \rho_m} V G^T + G V \frac{\partial G^T}{\partial \rho_m} + \Gamma_a \frac{\partial W_{a1}}{\partial \rho_m} \qquad \text{(F.74)}$$

$$\frac{\partial \dot{P}_3}{\partial \rho_m} = A_c \frac{\partial P_3}{\partial \rho_m} + \frac{\partial P_3}{\partial \rho_m} A_f{}^T + P_3 \frac{\partial A_f{}^T}{\partial \rho_m} + \frac{\partial K}{\partial \rho_m} C_a \left[P_2 - P_1 \right]$$

$$+ K C_a \left[\frac{\partial P_2}{\partial \rho_m} - \frac{\partial P_1}{\partial \rho_m} \right] \qquad \text{(F.75)}$$

$$\frac{\partial \dot{P}_4}{\partial \rho_m} = A_c \frac{\partial P_4}{\partial \rho_m} + \frac{\partial P_4}{\partial \rho_m} A_c^T + \frac{\partial K}{\partial \rho_m} C_a P_3^T + K C_a \frac{\partial P_3^T}{\partial \rho_m}$$

$$+ P_3 C_a^T \frac{\partial K^T}{\partial \rho_m} + \frac{\partial P_3}{\partial \rho_m} C_a^T K^T + \frac{\partial K}{\partial \rho_m} V K^T + K V \frac{\partial K^T}{\partial \rho_m}$$

$$(F.76)$$

$$\frac{\partial P_5}{\partial \rho_m} = \int_0^\tau e^{A_a \epsilon} \Gamma_a \frac{\partial W_{a1}}{\partial \rho_m} \Gamma_a^T e^{A_a^T \epsilon} d\epsilon \qquad (F.77)$$

$$\frac{\partial A_f}{\partial \rho_m} = - \frac{\partial G}{\partial \rho_m} C_a \qquad (F.78)$$

F.5   Gradient with Respect to W_1

Recall that W_1 is the covariance of the white noise component of the input disturbance. The equations for this case is the same as with respect to ρ_m except

$$\frac{\partial W_{a1}}{\partial W_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (F.79)$$

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Appendix G

Digital computer programs were developed to propagate the mean states and covariances given by the coupled matrix differential Eqs (5.4.24) to (5.4.26), (5.4.12), and (5.4.28) to (5.4.31). Also the gradient equations developed in Appendix F for calculation of $\partial J/\partial \phi$ are incorporated into the programs. These programs used the computational algorithms and subroutines developed by Kleinman for use in Linear System Studies (Ref 24). Figure G.1 shows a flow diagram for these programs.

Figure G.1

```
┌─────────────────────────────────┐
│    Set Values of Parameters     │
└─────────────────────────────────┘
                │
                ▼
┌─────────────────────────────────┐
│    Set Initial Conditions       │
│       Subroutine Init           │
└─────────────────────────────────┘
                │
                ▼
┌─────────────────────────────────┐
│ Calculate Angular Position Velocity │
│   and Acceleration for time t   │
│       Subroutine Angle          │
└─────────────────────────────────┘
                │
                ▼
┌─────────────────────────────────┐
│  Calculate Feedback Gains for time t │
│       Subroutine FBG1           │
└─────────────────────────────────┘
                │
                ▼
┌─────────────────────────────────┐
│ Propagate Coupled Matrix Differential │
│ Equations for Mean States, Covariances, │
│   Gradient Equations from time t │
│          to t + Δt              │
│                                 │
│  Subroutines Set, Step, F2, F3, │
│       Dell, and Grady.          │
└─────────────────────────────────┘
                │
                ▼
┌─────────────────────────────────┐
│  Calculate Current Value of Gradient │
│                                 │
└─────────────────────────────────┘
```

Calculate Current Value of Gradient

$$-\frac{\partial J}{\partial \phi} = -2 \sum_{i=1}^{n} \left[\frac{\partial y(t_i)}{\partial \phi}\right]^T \left[z(t_i) - y(t_i)\right]$$

B   A

B   A

Calculate Current Value of Cost

$$COST = \sum_{i=1}^{n} \left[z(t_i) - y(t_i) \right]^T \left[z(t_i) - y(t_i) \right]$$

Calculate $M(\phi) = \sum_{i=1}^{n} \left[\frac{\partial y(t_i)}{\partial \phi} \right]^T \left[\frac{\partial y(t_i)}{\partial \phi} \right]$

$t = t + \Delta t$

Adjust Noise Values $V(t)$ and $V_m(t)$.

No   Is $t > t_f$

Yes

Calculate $\left[M(\phi) \right]^{-1}$

Stop.

Adjust Parameter Values off Line Using $\partial J / \partial \phi$

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Bibliography

1. Abramson, Paul D. <u>Simultaneous Estimation of the State and Noise Statistics in Linear Dynamical Systems.</u> ScD Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts, May, 1968.

2. Apostol, T. M. <u>Mathematical Analysis, A Modern Approach to Advanced Calculus.</u> Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1957.

3. Aström, K. J. and P. Eykhoff. "System Identification-- A Survey." <u>Automatica, 7:</u> 123-162, 1971.

4. Banks, H. T. and G. M. Groome, Jr. "Convergence Theorems for Parameter Estimation by Quasilinearization." <u>Journal of Math. Analysis and Applications 42:91-109,</u> 1973.

5. Baron, S., D. L. Kleinman, D. C. Miller, W. H. Levison, J. I. Elkind. <u>Application of Optimal Control Theory to the Prediction of Human Performance in a Complex Task.</u> AFFDL-TR-69-81, March 1970.

6. Bellman, R. and K. J. Astrom. "On Structural Identifiability." <u>Math. Biosci. Vol. 7:329-339, 1970.</u>

7. Cramer, H. <u>Mathematical Methods of Statistics.</u> Princeton University Press, Princeton, New Jersey, 1946.

8. Denham, M. J. "Canonical Forms for the Identification of Multivariable Linear Systems." <u>IEEE Trans. Auto. Control, AC-19:646-656, Dec. 1974.</u>

9. Eykhoff, P. "Process Parameter and State Estimation." <u>Automatica 4:205-233, 1968.</u>

10. Gilbert, E. G. "Controllability and Observability in Multivariable Control Systems." <u>Journal of Society of Industrial Applied Math-Control Ser., Series A, Vol. I, No. 2:128-151, 1963.</u>

11. Glover, Keith and J. C. Willems. "Parametrizations of Linear Dynamical Systems: Cononical Forms and Identifiability." <u>IEEE Transactions on Automatic Control, Vol. AC-19, Dec. 1974.</u>

12. Grewal, M. S. and K. Glover. "Identifiability of Linear and Nonlinear Dynamical Systems." IEEE Transactions on Automatic Control, Vol. AC-21, Dec. 1976.

13. Griffith, E. W. and K. S. P. Kumar. "On the Observability of Nonlinear Systems: I." Journal of Mathematical Analysis and Applications 35:135-147, 1971.

14. Groome, G. M. Jr. Identification of Dynamical Systems. PhD Thesis, Brown University, 1972.

15. Hamton, R. L. T. and J. R. Cooke. "Unsupervised Tracking of Maneuvering Vehicles." IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-9, No. 2: March 1973.

16. Jazwinski, Andrew H. Stochastic Processes and Filtering Theory. Academic Press, New York and London, 1970.

17. Kalman, R. E., P. L. Falb, M. A. Arbib. Topics in Mathematical System Theory. McGraw-Hill, New York, 1969.

18. Kalman, R. E. "Canonical Structure of Linear Dynamical Systems." Proc. of National Academy of Science, Vol 48, No. 4:596-600, April 1962.

19. Kalman, R. E. "Mathematical Description of Linear Dynamical Systems." Journal of Society of Industrial Applied Mathematics, Control Ser. Series A, Vol. 1 No. 2:152-192, 1963.

20. Kleinman, D. L. A Predictive Model of the Human Operator in the Vulcan Air Defense System. Systems Control Inc., Cambridge, Massachusetts, November, 1972.

21. Kleinman, D. L. and S. Baron. Analytic Evaluation of Display Requirements for Approach to Landing. Bolt Beranek and Newman, Inc. Cambridge, Mass. NASA CR-1952, November 1971.

22. Kleinman, D. L. and S. Baron. Manned Vehicle Systems Analysis by Means of Modern Control Theory. Bolt Beranek and Newman, Inc. Cambridge, Mass. NASA CR-1753, June, 1971.

23. Kleinman, D. L. and T. R. Perkins. "Modeling Human Performance in a Time-Varying Anti Aircraft Tracking Loop." IEEE Transactions on Automatic Control, AC-19, August 1974.

24. Kleinman, D. L. Computer Programs Useful in Linear Systems Studies, SCI Technical Memorandum, Dec. 1971.

25. Kleinman, D. L. "Solving the Optimal Attention Allocation Problem in Manual Control." IEEE Transactions on Automatic Control, AC-21, December 1976.

26. Kleinman, D. L. "Optimal Control of Linear Systems with Time-Delay and Observation Noise." IEEE Transactions on Automatic Control, October 1969.

27. Kleinman, D. L., S. Baron, W. Levison. "A Control Theoretic Approach to Manned-Vehicle Systems Analysis." IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, December 1971.

28. Kostyukovskii, Yu M. L. "Observability of Nonlinear Controlled Systems." Automation and Remote Control 29: 1384-1396, 1968.

29. Kostyukovskii, Yu M. L. "Simple Conditions of Observability of Nonlinear Controlled Systems." Automation and Remote Control 29:1575-1584, 1968.

30. Kreindler, E. and P.E. Sarachik. "On the Concepts of Controllability and Observability of Linear Systems." IEEE Trans. on Auto. Control, Vol. AC-9, No. 2: 129-136, 1964.

31. Kwakernaak and Sivan. Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

32. Levison, N. H., S. Baron, and D. Kleinman. A Model for Human Controller Remnant. Bolt, Beranek, and Newman, Inc. Cambridge, Mass. Report 1731. October 1968.

33. Maybeck, Peter S. Combined State and Parameter Estimation for On-Line Applications, PhD Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Mass., Feb. 1972.

34. McAulay, R. J. and E. Denlinger. "A Decision-Directed Adaptive Tracker." IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-9, No. 2, March 1973.

35. Mehra, Raman K. "On the Identification of Variances and Adaptive Kalman Filtering." IEEE Transactions on Automatic Control, Vol. AC-15, April 1970.

36. Mehra, R. K. Identification in Control and Econometrics: Similarities and Differences. Tech. Rpt. No. 647. Div. Eng. and Applied Physics, Harvard University, Cambridge, Mass., July 1973.

37. Nering, Evar D. Linear Algebra and Matrix Theory.
    John Wiley and Sons, Inc., New York, 1970.

38. Ogata, K. State Space Analysis of Control Systems.
    Prentice-Hall, Inc., Englewood Cliffs, New Jersey,
    1967.

39. Ortega, J. M. and W. C. Rheinbolt. Iterative Solution
    of Nonlinear Equations in Several Variables. Academic
    Press, New York and London, 1970.

40. Phatak, Anil, H. Weinert, I. Segall. Identification of
    the Optimal Control Model for the Human Operator.
    Systems Control, 1974.

41. Phatak, Anil. Improvement in Weapon System Effective-
    ness by Application of Identification Methods for
    Determining Human Operator Performance Decrements Under
    Stress Conditions. Systems Control Inc., Palo Alto,
    California. AMRL-TR-73-38, December 1973.

42. Phillipson, G. A. Identification of Distributed Systems.
    Modern Analytical and Comp. Meth. in Science and Math.,
    R. Bellman, ed. American Elsevier Pub. Co., Inc.,
    New York, 1971.

43. Royden, H. L. Real Analysis. The Macmillan Co.,
    London, 1968.

44. Sage, A. P. and J. L. Melsa. System Identification.
    Academic Press, New York and London, 1971.

45. Sage, A. P. Optimal Systems Control. Prentice-Hall,
    Inc., Englewood Cliffs, New Jersey.

46. Sokolnikoff, I. S. Tensor Analysis. John Wiley & Sons,
    New York, 1967.

47. Sansone, G. and R. Conti. Nonlinear Differential
    Equations. The Macmillan Co., New York, 1964.

48. Stepner, David E. and R. K. Mehra. Maximum Likelihood
    Identification and Optimal Input Design for Identifying
    Aircraft Stability and Control Derivatives. Systems
    Control Inc., Palo Alto, Calif. NASA Report CR-2200,
    March 1973.

49. Taylor, Lawrence W. Jr. and K. Iliff. Systems Identi-
    fication Using a Modified Newton-Raphson Method--A
    Fortran Program, NASA Tech. Note TN D-6734, May 1972.

50. Tse, E. and J. Anton. "On Identifiability of Parameters." IEEE Transactions on Automatic Control, Vol. AC-17:637-646, October 1972.

51. Tse, E. and H. Weinert. "Correction and Extension of 'On the Identifiability of Parameters'." IEEE Transactions on Automatic Control, Vol. AC-18, 1973.

52. Tse, E. "Information Matrix and Local Identifiability of Parameters." Proceedings Joint Auto. Control Conf. Ohio State University, Columbus, Ohio, 1973.

53. Wald, A. "Note on the Consistency of the Maximum Likelihood Estimate." Ann. Math. Statistics, Vol. 20, 1949.

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Two very important and fundamental topics in the theory of dynamical systems are observability and identifiability. There has been limited work on the observability and identifiability of nonlinear dynamical systems. This problem is addressed by viewing the observation process as a nonlinear mapping of the initial states (and parameters) to the output measurements. The problem of observability and identifiability is then reduced to finding conditions under which the nonlinear mapping is one-to-one.

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Using such an approach, the theory of nonlinear functions is extended to find new sufficient conditions for local and global observability and identifiability for nonlinear dynamical systems. The nonlinear theory is then applied to the problem of determining the identifiability of the optimal control model for the human operator. The model assumes that the operator performance is dictated by the desire to behave optimally with respect to a chosen cost functional under constraints. Within the model structure are a number of parameters, the values of which determine the model response. Previous attempts to establish the identifiability of the model parameters have been limited to linear system theory. The identifiability of the model parameters is established using the previously developed nonlinear theory, a gradient computational technique is used to estimate model parameters, and the results of model response to experimental simulator data are presented.